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I N D I C E

EIGENVALUES OF THE p-LAPLACE OPERATOR WITH RESPECT TO TWO OBSTACLES

Nota di Annamaria Canino e Umile Perri
Presentata dal socio Carlo Sbordone
Adunanza del 12/1/91

Riassunto

Si studia una classe di disequazioni variazionali ottenuta considerando il problema degli autovalori del p-Laplaciano rispetto a due ostacoli. Utilizzando una teoria di punti critici per funzionali non regolari, si dimostra l'esistenza di infinite soluzioni per il problema considerato.

Abstract

We study a class of variational inequalities obtained by considering the problem of the eigenvalues of the p-Laplace operator with respect to two obstacles. By using a lower critical point theory for non regular functionals, we prove the existence of infinitely many solutions of the considered problem.

Keywords

p-Laplace operator, subdifferential function, Sobolev's immersion theorem, lower critical points, Z_2-category.

INTRODUCTION

The aim of this paper is to study the problem of the eigenvalues of the p-Laplace operator with respect to two obstacles, namely

$$(P) \qquad \begin{cases} u \in K_g \cap V \\ g(x,u)(v-u) \in L^1(\Omega) \quad \forall v \in K_g \\ \int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx + \\ +\lambda \int_\Omega \phi(x,u)(v-u)dx \geq 0 \qquad \forall v \in K_g \end{cases}$$

where

$$K_g = \left\{ v \in W_0^{1,p}(\Omega) : \psi_1 \leq \overline{v} \leq \psi_2 \ \ p-\text{cap. a.e.}, \ G(\cdot, v(\cdot)) \in L^1(\Omega) \right\},$$

$$V = \left\{ v \in W_0^{1,p}(\Omega) : \int_\Omega \Phi(x,v)dx = \rho \right\},$$

with Ω a bounded open subset of $\mathbb{R}^n$; $p > \dfrac{2n}{n+2}$; ψ_1, ψ_2 two assigned real measurable functions on Ω; g and ϕ two functions of Carathéodory type on $\Omega \times \mathbb{R}$ and $G(x,t) = \int_0^t g(x,s)ds$, $\Phi(x,t) = \int_0^t \phi(x,s)ds$.
We use methods of non smooth analysis as developed in [5],[6],[7],[8],[9],[10]. Indeed, the problem is reduced to prove existence and multiplicity of the lower critical points of the functional

$$f(u) = \frac{1}{p} \int_\Omega |Du|^p \, dx - \int_\Omega G(x,u) \, dx$$

on the constraint $X = K_g \cap V$.

To obtain the mutiplicity result, a critical point theory for non smooth functionals (see [8],[14],[15]) is applied. Let us point out the main difficulty. The functional f is defined in $W_0^{1,p}(\Omega)$ and also the constraint X is naturally endowed with the $W_0^{1,p}$-topology. On the contrary, if we apply critical point theory, we need a Hilbertian structure.
This difficulty has been overcome by finding, by means of Sobolev's fractional imbedding theorem (see [17]), a Hilbert space H such that $W_0^{1,p} \hookrightarrow H$ and by taking regularity and growth assumptions on ϕ and g suitable to study the variational problem with the topology of H. Moreover, let us note that our technique requires $p > \dfrac{2n}{n+2}$, because we need the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

We recall that in the case p=2, the problem (P) is studied in [2], [3] and [12]. In [2] and [3], the existence of infinitely many solutions of (P) has been obtained, considering the inequalities in K_g in almost everywhere sense. It is used the L^2−topology. In [12], results of the same kind have been obtained. But the $W_0^{1,2}$-topology is used and weaker hypotheses than in [2] and [3] are assumed. The inequalities in K_g are regarded in the $W_0^{1,2}$-capacity sense. This permits to consider thin obstacles.

The paper is divided as follows. In section 1, we recall some notions of non-smooth analysis. In section 2, we list the assumptions and we state the main results of this paper (theo.2.2 and theo 2.3). In section 3, we prove some properties of our functional, in particular that f is C(P,Q) (in the sense of [2]) and we give the proof of the theorem characterizing the solutions of (P) as the lower critical points of f (Theo.3.1). Section 4 is devoted to the study of the category of X and to the proof of the main theorems.

1. RECALLS OF NON-SMOOTH ANALYSIS

In this section, let us recall some notions of non-smooth analysis (cfr. [2], [3], [5], [6], [7], [9],[10]).

Let us denote by H a real Hilbert space, and by $\|\cdot\|$ and $<\cdot,\cdot>$ its norm and scalar product, respectively.

DEFINITION 1.1 — *(see, also, [2], [7], [9]). Let* $f : H \to \mathbb{R} \cup \{+\infty\}$ *be a map. We set* $D(f) = \{u \in H : f(u) < +\infty\}$. *Let* u *belong to* $D(f)$. *The function* f *is said to be subdifferentiable at* u *if there exists* $\alpha \in H$ *such that:*

$$\liminf_{v \to u} \frac{f(v) - f(u) - <\alpha, v-u>}{\|v-u\|} \geq 0.$$

We denote by $\partial^- f(u)$ *the (possibly empty) set of such* α*'s and we set*

$$D(\partial^- f) = \{u \in D(f) : \partial^- f(u) \neq \emptyset\}.$$

It is easy to check that $\partial^- f(u)$ *is convex and closed* $\forall\, u \in D(f)$; *if* $u \in D(\partial^- f)$, $grad^- f(u)$ *will denote the element of minimal norm of* $\partial^- f(u)$.

Moreover, let E *be a subset of* H. *We denote by* I_E *the function:*

$$I_E(u) = \begin{cases} 0 & u \in E \\ +\infty & u \in H \setminus E \end{cases}$$

It is easy to check that $\partial^- I_E(u)$ *is a cone* $\forall\, u \in E$.

We will call (outward) normal cone to E *at* u *the set* $\partial^- I_E(u)$.

DEFINITION 1.2 — *A point* $u \in D(f)$ *is said to be a lower critical point for* f *if* $0 \in \partial^- f(u)$; $c \in \mathbb{R}$ *is said to be a lower critical value for* f *if there exists* $u \in D(f)$ *such that* $0 \in \partial^- f(u)$ *and* $f(u) = c$.

DEFINITION 1.3 — *(see [5], [9]) Let* W *be an open subset of* H. *A function* $f : W \longrightarrow \mathbb{R} \cup +\infty$ *is said to have a* φ*-monotone subdifferential if there exists a continuous function*

$$\varphi : D(f) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$$

such that:

$$< \alpha - \beta, u - v > \geq -\left(\varphi(u, f(u), \|\alpha\|) + \varphi(v, f(v), \|\beta\|)\right) \|u - v\|^2$$

whenever

$$u, v \in D(\partial^- f), \ \alpha \in \partial^- f(u) \text{ and } \beta \in \partial^- f(v).$$

if $p \geq 1$, f is said to have a φ-monotone subdifferential of order p, if there exists a continuous function

$$\chi : D(f)^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$$

such that:

$$< \alpha - \beta, u - v > \geq -\chi(u, v, f(u), f(v))(1 + \|\alpha\|^p + \|\beta\|^p)\|u - v\|^2$$

whenever

$$u, v \in D(\partial^- f), \ \alpha \in \partial^- f(u) \text{ and } \beta \in \partial^- f(v).$$

Finally, we conclude this section recalling the definition of function of class C(P,Q).

DEFINITION 1.4 — *(see [2] and [12]). Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a map. Let P and Q be real continuous functions on $D(f) \times D(f)$. f is said to belong to class $C(P,Q)$ if $\forall u \in D(f)$ such that $\partial^- f(u) \neq \emptyset$ and $\forall \alpha \in \partial^- f(u)$ it results*

$$f(v) \geq f(u) + < \alpha, v - u > - \left[P(u, v)\|\alpha\| + Q(u, v)\right] \|v - u\|^2 \ \forall v \in D(f).$$

REMARK 1.5 — *If f is a map belonging to class $C(P,Q)$, then f has a φ-monotone subdifferential of order 2.*

2. THE FRAMEWORK AND THE MAIN RESULTS

Let Ω be a bounded open set of $\mathbb{R}^n$, with $n > 2$.

Let us consider the functions : $G : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $G(x, \cdot)$is of class C^1 for almost all x and $G(\cdot, t)$ is measurable for all t; $\Phi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\Phi(x, \cdot)$ is of class C^2 for almost all x and $\Phi(\cdot, t)$ is measurable for all t; $\psi_1, \psi_2 : \Omega \longrightarrow \mathbb{R}$ Borel measurable with $\psi_1 \leq \psi_2$ a.e. in Ω. Moreover, let us set $D_t G(x, t) = g(x, t)$ and $D_t \Phi(x, t) = \phi(x, t)$.

Now, let us fix a real number $p > \frac{2n}{n+2}$.

In the following, we will consider the Sobolev space $W_0^{1,p}(\Omega)$ and we will denote by H, the Hilbert space $W_0^{s,2}(\Omega)$, where

$$s = \begin{cases} \frac{n(p-2)+2p}{2p} & \frac{2n}{n+2} < p < 2 \\ 1 & p \geq 2 \end{cases}$$

Let us recall that $W_0^{1,p}(\Omega) \hookrightarrow W_0^{s,2}(\Omega)$ (see [17]).

$\|\cdot\|$ and $< \cdot, \cdot >$, will be the norm and the scalar product of H, respectively.

Let us set

$$V = \left\{ v \in H : \int_\Omega \Phi(x, v)dx = \rho \right\} (\rho \neq 0);$$

$$K = \left\{ v \in W_0^{1,p}(\Omega) : \psi_1 \leq \overline{v} \leq \psi_2 \ \ p-\text{cap-a.e. on } \Omega \right\};$$

$$K_g = \left\{ v \in K : G(\cdot, v(\cdot)) \in L^1(\Omega) \right\};$$

$$u_K = \psi_1^+ - \psi_2^-;$$

(We denote by u^+ the positive part of u i.e. $u^+(x) = u(x)$ if $u(x) \geq 0$ and $u^+(x) = 0$ if $u(x) < 0$; $u^- = (-u)^+$).

Let us make the following assumptions on the functions G, Φ, ψ_1, ψ_2.

Given q, with $2 \leq q < \frac{np}{n-p}$ if $p < n$ and $2 \leq q < +\infty$ if $p \geq n$,

(G.1) there exist $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbb{R}$ such that

$$G(x, t) \leq a_0(x) + b_0|t|^q;$$

(G.2) there exist $a_1 \in L^r(\Omega)$ with

$$r = \begin{cases} \frac{q}{q-2} & \frac{2n}{n+2} < p < 2 \\ \frac{n}{2} & p \geq 2 \end{cases}$$

(with the convention that, if $q = 2$ and $\frac{2n}{n+2} < p < 2$, then $r = \infty$), $b_1 \in \mathbb{R}$ and a Carathéodory function $w : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$0 \leq w(x, t_1, t_2) \leq a_1(x) + b_1 \left[|t_1|^{q-2} + |t_2|^{q-2} \right]$$

and such that

$$G(x,t_2) \le G(x,t_1) + g(x,t_1)(t_2 - t_1) + w(x,t_1,t_2)|t_2 - t_1|^2;$$

(G.3) $G(\cdot,t) \in L^1(\Omega)$, $\forall t \in \mathbb{R}$.

Given $\widetilde{q}$, with

$$\begin{cases} 2 \le \widetilde{q} \le \dfrac{np}{n-p} & \dfrac{2n}{n+2} < p < 2 \\ 2 \le \widetilde{q} \le \dfrac{2n}{n-2} & 2 \le p < n \\ 2 \le \widetilde{q} < +\infty & p \ge n \end{cases}$$

(Φ.1) there exist $c_0 \in L^1(\Omega), d_0 \in \mathbb{R}$, such that

$$|\Phi(x,t)| \le c_0(x) + d_0|t|^{\widetilde{q}};$$

(Φ.2) there exist $c_1 \in L^{\widetilde{q}'}(\Omega)$ $\left(\widetilde{q}' = \dfrac{\widetilde{q}}{\widetilde{q}-1}\right)$, $d_1 \in \mathbb{R}$ such that

$$|\Phi_t(x,t)| \le c_1(x) + d_1|t|^{\widetilde{q}-1};$$

(Φ.3) there exist $c_2 \in L^{\sigma}(\Omega)$, $\left(\sigma = (\dfrac{\widetilde{q}}{2})' = \dfrac{\widetilde{q}}{\widetilde{q}-2}\right)$, (with the convention that, if $\widetilde{q} = 2$, then $\sigma = \infty$), $d_2 \in \mathbb{R}$ such that

$$|\Phi_{tt}(x,t)| \le c_2(x) + d_2|t|^{\widetilde{q}-2};$$

(Φ.4) $\phi(x,t)t > 0 \quad \forall t \ne 0$

(without loss of generality, in the following we will suppose $\Phi(x,0) = 0$ a.e. in Ω);

(Ψ) $\psi_1, \psi_2 \in L^r(\Omega)$ where

$$r = \begin{cases} q & \dfrac{2n}{n+2} < p < 2 \\ q & p \ge 2, 2 \le q \le \dfrac{2n}{n-2} \\ \dfrac{n(q-2)}{2} & p > 2, q > \dfrac{2n}{n-2} \end{cases}$$

The aim of this paper is to study the following problem:

(P) To find a real number λ and a function u on Ω such that:

$$\begin{cases} u \in K_g \cap V \\ g(x,u)(v-u) \in L^1(\Omega) \quad \forall v \in K_g \\ \int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx + \\ +\lambda \int_\Omega \phi(x,u)(v-u)dx \ge 0 \qquad \forall v \in K_g \end{cases}$$

Now, after recalling a definition, we can state the main results of this paper.

DEFINITION 2.1 — *(see [2]). Let A and B be two subsets of a Hilbert space H; A and B are (externally)tangent at $u \in A \cap B$ if*

$$(-\partial^- I_A(u)) \cap \partial^- I_B(u) \neq \{0\}.$$

THEOREM 2.2 — *Let $K_g \cap V \neq \emptyset$ and K_g and V be not tangent at any point. Let us suppose that $(G.1), (G.2), (\Phi.1), (\Phi.2), (\Psi)$ hold with q such that*

$$2 \leq q < \frac{np}{n-p} \text{ if } p < n \text{ and } 2 \leq q < +\infty \text{ if } p \geq n,$$

and $\widetilde{q}$ such that

$$\begin{cases} 2 \leq \widetilde{q} \leq \dfrac{np}{n-p} & \dfrac{2n}{n+2} < p < 2 \\ 2 \leq \widetilde{q} \leq \dfrac{2n}{n-2} & 2 \leq p < n \\ 2 \leq \widetilde{q} < +\infty & p \geq n \end{cases}$$

Then, there exist $u \in K_g \cap V$ and $\lambda \in \mathbb{R}$ solving the problem (P).

THEOREM 2.3 — *Let $K_g \cap V \neq \emptyset$ and K_g and V be not tangent at any point. Let us suppose that $(G.1)-(G.3), (\Phi.1)-(\Phi.4)$ and (Ψ) hold, with q and $\widetilde{q}$ as in Theo.2.2. Moreover, let us assume that $g(x,t)$ and $\phi(x,t)$ are odd with respect to the variable t and $\psi = \psi_2 = -\psi_1 \geq 0$.*

Then, there exist a sequence $(u_k)_k \subset K_g \cap V$ and a real sequence $(\lambda_k)_k$ such that $\forall k \in N$, (u_k, λ_k) and $(-u_k, \lambda_k)$ are solutions of (P) and $\inf\{\lambda_k : k \in N\} = -\infty$.

3. VARIATIONAL CHARACTERIZATION OF THE SOLUTIONS OF THE PROBLEM

In this section, our aim is to prove that the solutions of the problem (P) can be characterized as lower critical points of a certain functional on a suitable space.

So, let us define the functionals: $f_1, f_0, f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ in the following way:

$$f_1(u) = \begin{cases} \frac{1}{p}\int_\Omega |Du|^p dx - \int_\Omega G(x,u)dx & u \in W_0^{1,p}(\Omega) \\ +\infty & u \in H \backslash W_0^{1,p}(\Omega) \end{cases}$$

$$f_0 = f_1 + I_K$$

$$f = f_0 + I_V.$$

Let us state the mentioned characterization.

THEOREM 3.1 — *Let us assume that* $(G.1), (G.2)$ *hold and let* u *belong to* $W_0^{1,p}(\Omega)$. *If there exists* $\lambda \in \mathbb{R}$ *such that* (u, λ) *is a solution of (P), then*

$$f(u) < +\infty \quad \text{and} \quad 0 \in \partial^- f(u).$$

The converse is true if K_g *and* V *are not tangent in* u.

In order to prove this theorem, we study the properties of the three previous functionals.

Let us start with f_1.

REMARK 3.2 —

a) *Under the assumption* $(G.1)$, *we have:*

a_1) f_1 *is well defined and*

$$D(f_1) = \left\{ u \in W_0^{1,p}(\Omega) : \ G(\cdot, u(\cdot)) \in L^1(\Omega) \right\}$$

a_2) f_1 *is lower semicontinuous if* $\dfrac{2n}{n+2} < p < 2$ or $p \geq 2$ and $2 \leq q \leq \dfrac{2n}{n-2}$.

b) *Under the assumptions* $(G.1)$ *and* $(G.2)$, *the function* $g(x, u)(v - u)$ *is lower semi-integrable* $\forall u \in W_0^{1,p}(\Omega)$ *and* $\forall v \in D(f_1)$.

c) *Under the assumptions* $(G.1) - (G.3)$, *the function* $g(x, u)u$ *is upper semi-integrable* $\forall u \in W_0^{1,p}(\Omega)$.

d) *Under the assumption* $(G.2), \forall t_1, t_2 \in \mathbb{R}$ *and* $s \in [0, 1]$, *we have:*

$$s\, G(x, t_1) + (1 - s)\, G(x, t_2) \leq$$

$$\leq G(x, s\, t_1 + (1 - s) t_2)) + s(1 - s) \left[a_1(x) + b_1 |t_1|^{q-2} + b_1 |t_2|^{q-2} \right] |t_2 - t_1|^2.$$

In the following, if $p < 2$, we will use the convention:

$$|Du|^{p-2} Du = 0 \qquad \text{if} \qquad Du = 0.$$

PROPOSITION 3.3 — *Under the assumptions* $(G.1)$ *and* $(G.2)$, *we have:*

a) $D(f_1)$ *is convex. Moreover, if* $w \in L^q(\Omega)$ *and there exist* $u, v \in L^q(\Omega)$ *such that* $G(\cdot, u), G(\cdot, v) \in L^1(\Omega)$ *and* $u \wedge v \leq w \leq u \vee v$ *a.e. in* Ω, *then* $G(\cdot, w) \in L^1(\Omega)$.

b) *If* u *and* v *belong to* $D(f_1)$, *then*

b_1) there exists $c = c(b_1, n, q, \Omega)$ such that

$$f_1(v) \geq f_1(u) + \int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx +$$

$$-c\left\{\|a_1\|_{L^r(\Omega)} + \|u\|^{q-2}_{W_0^{1,p}(\Omega)} + \|v\|^{q-2}_{W_0^{1,p}(\Omega)}\right\} \|v-u\|^2_{W_0^{1,p}(\Omega)}$$

b_2)

$$\lim_{t\to 0^+} \frac{f_1(u+t(v-u)) - f_1(u)}{t} =$$

$$= \int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx.$$

c) If u and v belong to $D(f_1)$ and $\partial^- f_1(u) \neq \emptyset$, then: the function $g(x,u)(v-u)$ is integrable.

d) If $u \in D(f_1)$ and $\alpha \in H$, then: $\alpha \in \partial^- f_1(u)$ if and only if $\forall v \in D(f_1)$

$$\int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx \geq < \alpha, v-u > .$$

e) If $\frac{2n}{n+2} < p < 2$ or $p \geq 2, 2 \leq q \leq \frac{2n}{n-2}$, $u \in D(f_1)$ and $\alpha \in \partial^- f_1(u)$, then: $\forall v \in D(f_1)$

$$f_1(v) \geq f_1(u) + < \alpha, v-u > -c\left\{\|a_1\|_{L^r(\Omega)} + \|u\|^{q-2} + \|v\|^{q-2}\right\} \|v-u\|^2$$

i.e. $f_1 \in C(0,Q)$.

Proof. - a) Let u, v be in $D(f_1)$and t in $[0,1]$. By definition of f_1, we have

$$f_1(u+t(v-u)) = \frac{1}{p}\int_\Omega |D(u+t(v-u))|^p dx - \int_\Omega G(x, u+t(v-u))dx,$$

and by Remark 3.2 d)

$$\int_\Omega G(x,(1-t)u+tv)dx > -\infty.$$

b_1) Since the real map $|\cdot|^p$ $(p>1)$ defined on $\mathbb{R}^n$ is convex and continuously differentiable, then:

$$f_1(v) - f_1(u) = \frac{1}{p}\int_\Omega |Dv|^p dx - \int_\Omega G(x,v)dx - \frac{1}{p}\int_\Omega |Du|^p dx + \int_\Omega G(x,u)dx \geq$$

$$\geq \int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx - \left\{\int_\Omega a_1(x)|v-u|^2 dx + \right.$$

$$+b_1\left[\int_\Omega |u|^{q-2}|v-u|^2dx+\int_\Omega |v|^{q-2}|v-u|^2dx\right]\Big\}. \qquad (3.3.1)$$

By applying Hölder's inequality to (3.3.1), in the case $\frac{2n}{n+2}<p<2$ and $p\geq 2$, $q>\frac{2n}{n-2}$, we have

$$f_1(v)\geq f_1(u)+\int_\Omega |Du|^{p-2}DuD(v-u)dx-\int_\Omega g(x,u)(v-u)dx+$$

$$-\left\{\|a_1\|_{L^r(\Omega)}+b_1\|u\|^{q-2}_{L^q(\Omega)}+b_1\|v\|^{q-2}_{L^q(\Omega)}\right\}\|v-u\|^2_{L^q(\Omega)} \qquad (3.3.2)$$

and in the case $p\geq 2$, $q\leq\frac{2n}{n-2}$,

$$f_1(v)\geq f_1(u)+\int_\Omega |Du|^{p-2}DuD(v-u)dx-\int_\Omega g(x,u)(v-u)dx+$$

$$-\left\{\|a_1\|_{L^r(\Omega)}+b_1\|u\|^{q-2}_{L^q(\Omega)}+b_1\|v\|^{q-2}_{L^q(\Omega)}\right\}\|v-u\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \qquad (3.3.3).$$

By applying Sobolev's imbedding theorem to (3.3.2) and (3.3.3), we get the thesis.

b_2) By b_1), we have:

$$\liminf_{t\to 0^+}\frac{f_1(u+t(v-u))-f_1(u)}{t}\geq$$

$$\geq\int_\Omega |Du|^{p-2}DuD(v-u)dx-\int_\Omega g(x,u)(v-u)dx.$$

On the other hand, by definition of f_1, we have:

$$\frac{f_1(u+t(v-u))-f_1(u)}{t}=\frac{1}{t}\left\{\frac{1}{p}\int_\Omega |D(u+t(v-u))|^p dx-\frac{1}{p}\int_\Omega |Du|^p dx+\right.$$

$$\left.-\int_\Omega [G(x,u+t(v-u))-G(x,u)]\,dx\right\}\quad \forall t\in(0,1].$$

Now, let us define the map $h:[0,1]\longrightarrow\mathbb{R}$ in the following way

$$h(t)=\int_\Omega |Du+tD(v-u)|^p dx.$$

Since $h(0)=\int_\Omega |Du|^p dx$ and $h'(0)=p\int_\Omega |Du|^{p-2}DuD(v-u)dx$, by Taylor's formula, we have:

$$\frac{f_1(u+t(v-u))-f_1(u)}{t}=\frac{1}{t}\left\{\frac{1}{p}\left[tp\int_\Omega |Du|^{p-2}DuD(v-u)dx+o(t)\right]+\right.$$

$$- \int_\Omega [G(x, u + t(v-u)) - G(x,u)]\, dx \Big\} .$$

Now, by Remark 3.2 d) and Fatou's lemma,

$$\liminf_{t\to 0+} \int_\Omega \frac{G(x, u + t(v-u)) - G(x,u)}{t} dx \geq \int_\Omega g(x,u)(v-u)dx.$$

So

$$\limsup_{t\to 0^+} \frac{f_1(u+t(v-u)) - f_1(u)}{t} \leq$$

$$\leq \int_\Omega |Du|^{p-2} Du D(v-u)dx - \int_\Omega g(x,u)(v-u)dx.$$

and the thesis is proved.

c) By Remark 3.2 b), it is enough to prove that

$$\int_\Omega g(x,u)(v-u)dx < +\infty.$$

Let $\alpha \in \partial^- f_1(u)$, then by b_2), we have

$$< \alpha, v-u > \leq \int_\Omega |Du|^{p-2} Du D(v-u)dx - \int_\Omega g(x,u)(v-u)dx$$

and this gives the thesis.

d) Let $\alpha \in \partial^- f_1(u)$, then, by b_2), the thesis follows. The converse is trivial.

e) We proceed as in the proof of b_1). By applying Hölder's inequality to (3.3.1) and recalling that $H \hookrightarrow L^q(\Omega)$, we have:

$$f_1(v) \geq f_1(u) + \int_\Omega |Du|^{p-2} Du D(v-u)dx - \int_\Omega g(x,u)(v-u)dx +$$

$$- c \left\{ \|a_1\|_{L^r(\Omega)} + \|u\|^{q-2} + \|v\|^{q-2} \right\} \|v-u\|^2 .$$

Now, it is enough to apply d) to this last inequality. □

THEOREM 3.4 — *Under the assumptions (G.1)-(G.3), we have:*

a) $W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \subset D(f_1)$;

b) if $u \in D(f_1)$ *and* $\partial^- f_1(u) \neq \emptyset$ *then:*

b_1) $g(x,u)v$ *is integrable* $\forall v \in D(f_1)$;

b_2) $g(x,u)v$ *is lower integrable* $\forall v \in L^q(\Omega)$ *such that* $G(x,v)$ *is integrable;*

c) if $u \in D(f_1)$ *and* $\alpha \in \partial^- f_1(u)$ *then:*

(c_1)

$$\int_\Omega |Du|^{p-2}DuDvdx - \int_\Omega g(x,u)vdx \geq < \alpha, v > \quad \forall v \in D(f_1);$$

(c_2)

$$\lim_{t\to 0^+} \frac{f_1(u+t(v-u)) - f_1(u)}{t} \geq < \alpha, v-u > \quad \forall v \in D(f_1);$$

d) if $u \in D(f_1)$ and $\alpha \in H$ then:
$\alpha \in \partial^- f_1(u)$ if and only if $\forall v \in D(f_1)$

$$\int_\Omega |Du|^{p-2}DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx \geq < \alpha, v-u > .$$

PROOF. - a) If $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, by Remark 3.2 d),

$$\int_\Omega G(x,u)dx > -\infty.$$

$b_1)$ It is enough to apply Prop.3.3 c).

$b_2)$ If $v \in L^q(\Omega)$ and $G(x,v) \in L^1(\Omega)$, by $(G.2)$, we have

$$g(x,u)v \geq G(x,v) - G(x,u) + g(x,u)u - w(x,u,v)|v-u|^2 \geq$$

$$\geq G(x,v) - G(x,u) + g(x,u)u +$$

$$-c\left[\|a_1\|_{L^r(\Omega)} + \|u\|^{q-2}_{W_0^{1,p}(\Omega)} + \|v\|^{q-2}_{W_0^{1,p}(\Omega)}\right] \|v-u\|^2_{W_0^{1,p}(\Omega)}.$$

Since $g(x,u)u$ is integrable by Prop.3.3 c), then $\int_\Omega g(x,u)vdx > -\infty$.

$c_1)$ Let us start proving that: $\quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega |Du|^{p-2}DuDvdx - \int_\Omega g(x,u)vdx = < \alpha, v > . \tag{3.4.1}$$

We observe that (3.4.1) is true $\forall v \in C_0^\infty(\Omega)$. Indeed, if $v \in C_0^\infty(\Omega)$ by Prop.3.3 d), we have:

$$\int_\Omega |Du|^{p-2}DuDv\,dx - \int_\Omega g(x,u)v\,dx - < \alpha, v > \geq$$

$$\geq \int_\Omega |Du|^p dx - \int_\Omega g(x,u)u\,dx - < \alpha, u > . \tag{3.4.2}$$

Setting $v_t = tv$ for $t \in \mathbb{R}$ and replacing in (3.4.2) v by v_t, we have:

$$t\left[\int_\Omega |Du|^{p-2}DuDvdx \; - \int_\Omega g(x,u)v\,dx - < \alpha, v >\right] \geq$$

$$\geq \int_\Omega |Du|^p dx - \int_\Omega g(x,u)u\,dx - <\alpha, u> .$$

Since t is an arbitrary real number then,

$$\int_\Omega |Du|^{p-2} DuDv\,dx - \int_\Omega g(x,u)v\,dx - <\alpha, v>= 0.$$

Now, let us consider $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, so there exists a sequence $(w_h)_h \subset C_0^\infty(\Omega)$ such that $\lim_{h\to\infty} w_h = w$ in $W_0^{1,p}(\Omega)$ and $\|w_h\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\Omega)}$. Then,

$$\int_\Omega |Du|^{p-2} DuDw_h dx - \int_\Omega g(x,u)w_h dx =< \alpha, w_h >$$

and by Lebesgue's Theorem, (3.4.1) follows.

Now, let us consider $v \in D(f_1)$ and let us set $v_k = (v \wedge k) \vee (-k)$. By applying (3.4.1) to v_k we have:

$$\int_\Omega |Du|^{p-2} DuDv_k dx - \int_\Omega g(x,u)v_k dx =< \alpha, v_k >$$

and by Fatou's lemma

$$\liminf_{k\to\infty} \int_\Omega g(x,u)v_k dx \geq \int_\Omega g(x,u)v\,dx$$

so, the thesis follows.

c_2) By Prop.3.3 b_2), we have

$$\lim_{t\to 0^+} \frac{f_1(u+t(v-u)) - f_1(u)}{t} =$$

$$\int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx$$

and by c_1), we get the thesis.

d) It follows by Prop.3.3d). □

PROPOSITION 3.5 —

a) *Under the assumptions* $(G.1), (G.2)$ *in the case* $\frac{2n}{n+2} < p < 2$ *or* $p \geq 2$, $2 \leq q \leq \frac{2n}{n-2}$ *and* $(G.1), (G.2), (\Psi)$ *in the case* $p > 2$, $q > \frac{2n}{n-2}$, *we have:*

a_1) f_0 *is lower semicontinuous on* H.

a_2) $f_0 \in C(0,Q)$. *Moreover, let us consider two sequences*

$(u_m)_m$, $(\alpha_m)_m \subset H$, with $u_m \in D(f_0)$, $\alpha_m \in \partial^- f_0(u_m)$, such that $(u_m)_m$ converges weakly to u in $W_0^{1,p}(\Omega)$ and $(\alpha_m)_m$ converges strongly to α in H. Then, $\alpha \in \partial^- f_0(u)$.

a_3) *If $u, v \in D(f_0)$ and $\partial^- f_0(u) \neq \emptyset$, then: $g(x,u)(v-u) \in L^1(\Omega)$.*

a_4) *If $u \in D(f_0)$ and $\alpha \in H$, then $\alpha \in \partial^- f_0(u)$ if and only if*

$$\int_\Omega |Du|^{p-2} Du D(v-u)dx - \int_\Omega g(x,u)(v-u)dx \geq < \alpha, v-u > \ \forall v \in D(f_0).$$

b) Under the assumptions (G.1)-(G.3), we have:

b_1) *$D(f_0) \neq \emptyset$ if and only if $D(f_0)$ is dense in K, with respect to the topology of H.*

b_2) *If $D(f_0) \neq \emptyset$, then:*

$$\psi_1^+, \psi_2^- \in L^q(\Omega) \text{ and } G(\cdot, \psi_1^+(\cdot)), G(\cdot, \psi_2^-(\cdot)) \in L^1(\Omega).$$

PROOF. - a_1) If $\frac{2n}{n+2} < p < 2$, or $p \geq 2, 2 \leq q \leq \frac{2n}{n-2}$, the statement follows by Remark 3.2 a_2).

In the case $p > 2, q > \frac{2n}{n-2}$, let us prove that $\forall c \in \mathbb{R}$ the set

$$F_c = \left\{ u \in W_0^{1,2}(\Omega) : f_0(u) \leq c \right\}$$

is closed in $W_0^{1,2}(\Omega)$.

To this aim, let us take $c \in \mathbb{R}$ and a sequence $(v_h)_h \subset F_c$ such that

$$\lim_{h \to \infty} v_h = v \text{ in } W_0^{1,2}(\Omega).$$

First of all, we will prove that $(v_h)_h$ is bounded in $W_0^{1,p}(\Omega)$.
Since, $(v_h)_h \subset D(f_0)$, by $(G.1)$, we have

$$c \geq f_0(v_h) = \frac{1}{p} \|v_h\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega G(x, v_h)dx \geq$$

$$\geq \frac{1}{p} \|v_h\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega a_0(x)dx - b_0 \|v_h\|^q_{L^q(\Omega)}.$$

Moreover, the hypothesis (Ψ) implies:

$$\|v\|_{L^q(\Omega)} \leq \|v\|_{L^r(\Omega)} \leq c \, \|\widetilde{\psi}\|_{L^r(\Omega)} \quad \forall v \in K, \tag{3.5.1}$$

where $\widetilde{\psi} = \max(|\psi_1|, |\psi_2|)$ and c depends only on n and Ω.

Thus, $\|v_h\|_{W_0^{1,p}(\Omega)}$ is bounded and we can consider a subsequence of $(v_h)_h$, still denoted by $(v_h)_h$, such that $(v_h)_h$ converges weakly in $W_0^{1,p}(\Omega)$ to v when $h \to \infty$. By Rellich's theorem $(v_h)_h$ converges strongly to v in $L^q(\Omega)$.
Moreover, since by $(G.1)$, the map $u \longrightarrow \int_\Omega G(x,u)dx$, is upper semicontinuous on $L^q(\Omega)$, we have:

$$c \geq \liminf_{h\to\infty} \left\{ \frac{1}{p}\|v_h\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega G(x,v_h)dx \right\} \geq$$

$$\geq \frac{1}{p}\|v\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega G(x,v)dx = f_0(v) \tag{3.5.2}$$

and then $v \in F_c$.

a_2) Let us consider $u, v \in D(f_0)$ and observe that if $\alpha \in \partial^- f_0(u)$ then

$$\liminf_{t\to 0^+} \frac{f_1(u+t(v-u)) - f_1(u)}{t} \geq < \alpha, v-u > \tag{3.5.3}$$

Thus, by Prop.3.3 b_2) and (3.3.1), we have

$$f_0(v) \geq f_0(u) + < \alpha, v-u > +$$

$$-\left\{ \int_\Omega a_1(x)|v-u|^2\,dx + b_1 \left[\int_\Omega |u|^{q-2}|v-u|^2\,dx + \int_\Omega |v|^{q-2}|v-u|^2\,dx \right] \right\} \tag{3.5.4}$$

Let us consider the case $\frac{2n}{n+2} < p < 2$.
By applying Hölder's inequality to (3.5.4), we get

$$f_0(v) \geq f_0(u) + < \alpha, v-u > +$$

$$-\left\{ \|a_1\|_{L^r(\Omega)} + b_1\|u\|^{q-2}_{L^q(\Omega)} + b_1\|v\|^{q-2}_{L^q(\Omega)} \right\} \|v-u\|^2_{L^q(\Omega)} \tag{3.5.5}$$

and recalling that $W_0^{s,2}(\Omega) \hookrightarrow L^t(\Omega)$ with $t \leq \frac{2n}{n-2s}$, we conclude that

$$f_0(v) \geq f_0(u) + < \alpha, v-u > +$$

$$-c\left\{ \|a_1\|_{L^r(\Omega)} + \|u\|^{q-2} + \|v\|^{q-2} \right\} \|v-u\|^2 \tag{3.5.6}$$

and this proves the first statement.

To prove the second statement, let us consider two sequences $(u_m)_m \subset D(f_0)$ and $(\alpha_m)_m \subset \partial^- f_0(u_m)$ such that $(u_m)_m$ converges weakly to u in $W_0^{1,p}(\Omega)$ and $(\alpha_m)_m$ converges strongly to α in H.

It is sufficient to prove that

$$\liminf_{v\to u} \frac{f_0(v) - f_0(u) - < \alpha, v-u >}{\|v-u\|} \geq 0 \tag{3.5.7}$$

From (3.5.5), we have

$$f_0(v) \geq f_0(u_m) + < \alpha_m, v - u_m > +$$

$$-\left\{\|a_1\|_{L^r(\Omega)} + b_1\|u_m\|^{q-2}_{L^q(\Omega)} + b_1\|v\|^{q-2}_{L^q(\Omega)}\right\}\|v-u_m\|^2_{L^q(\Omega)}. \tag{3.5.8}$$

By Rellich's theorem $u_m \to u$ in $L^q(\Omega)$ and as in (3.5.2), we have:

$$\liminf_{m\to\infty} f_0(u_m) \geq f_0(u).$$

So, by letting $m \to \infty$ in (3.5.8) and recalling that $H \hookrightarrow L^q(\Omega)$, (3.5.7) follows.

Now, let us consider the case $p \geq 2$. If $2 \leq q \leq \frac{2n}{n-2}$, by applying Hölder's inequality to (3.5.4), we have

$$f_0(v) \geq f_0(u) + < \alpha, v - u > +$$

$$-\|a_1\|_{L^r(\Omega)}\|v-u\|^2_{L^{\frac{2n}{n-2}}(\Omega)} - \left\{b_1\|u\|^{q-2}_{L^q(\Omega)} + b_1\|v\|^{q-2}_{L^q(\Omega)}\right\}\|v-u\|^2_{L^q(\Omega)}. \tag{3.5.9}$$

Instead if $u, v \in K$ and $q > \frac{2n}{n-2}$, by (Ψ), we know that $u, v \in L^{\frac{n(q-2)}{2}}(\Omega)$ and $|v-u|^2 \in L^{\frac{n}{n-2}}(\Omega)$. So, by applying, again, Hölder's inequality to (3.5.4), we obtain

$$f_0(v) \geq f_0(u) + < \alpha, v - u > +$$

$$-\left\{\|a_1\|_{L^r(\Omega)} + b_1\|u\|^{q-2}_{L^{\frac{n(q-2)}{2}}(\Omega)} + b_1\|v\|^{q-2}_{L^{\frac{n(q-2)}{2}}(\Omega)}\right\}\|v-u\|^2_{L^{\frac{2n}{n-2}}(\Omega)}. \tag{3.5.10}$$

By applying Sobolev's imbedding theorem to (3.5.9) and (3.5.10), we have

$$f_0(v) \geq f_0(u) + < \alpha, v - u > +$$

$$-c\left\{\|a_1\|_{L^r(\Omega)} + \|u\|^{q-2}_{L^r(\Omega)} + \|v\|^{q-2}_{L^r(\Omega)}\right\}\|v-u\|^2_{W_0^{1,2}(\Omega)}. \tag{3.5.11}$$

Thus, if $2 \leq q \leq \frac{2n}{n-2}$, (3.5.11) gives $f_0 \in C(0, Q)$. Instead if $q > \frac{2n}{n-2}$, since $u, v \in K$, it is enough to apply (3.5.1) to (3.5.11).

To prove the second statement, for $p \geq 2$, we can proceed as in the previous case, observing that, by (3.5.11), we have

$$f_0(v) \geq f_0(u_m) + < \alpha, v - u_m > +$$

$$-c\left\{\|a_1\|_{L^r(\Omega)} + \|u_m\|^{q-2}_{L^r(\Omega)} + \|v\|^{q-2}_{L^r(\Omega)}\right\}\|v-u_m\|^2_{W_0^{1,2}(\Omega)}. \tag{3.5.12}$$

Then, from (3.5.12) it is easy to get the thesis by Sobolev's imbedding theorem, in the case $2 \le q \le \frac{2n}{n-2}$ and by using (3.5.1) if $q > \frac{2n}{n-2}$.

a_3) By Remark 3.2 b) $g(x,u)(v-u)$ is lower semi-integrable.
On the other hand, by Prop. 3.3 b_2), and (3.5.3), we have

$$\int_\Omega g(x,u)(v-u)dx =$$

$$= \int_\Omega |Du|^{p-2}DuD(v-u)dx - \lim_{t\to 0^+}\frac{f_1(u+t(v-u))-f_1(u)}{t} \le$$

$$\le \int_\Omega |Du|^{p-2}DuD(v-u)dx - <\alpha, v-u> .$$

Then we get

$$\int_\Omega g(x,u)(v-u)dx < +\infty.$$

a_4) it follows by Prop.3.3 b_2).

b_1) Let us consider $u \in K$ and a sequence $(u_h)_h \subset W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $u_h \to u$ in H. Let us set $v_h = (u_h \vee \psi_1)\wedge(\psi_2)$.
Since $u_h \in D(f_1)$ we have $u_h \wedge \psi_2 \le v_h \le u_h \vee \psi_1$.
Then $v_h \in D(f_1)$. It is easy to see that $v_h \to u$ in H.

b_2) Let us consider $u \in D(f_0)$, then:
1) $0 \le \psi_1^+ \le |u|$ a.e. in Ω and
2) $0 \le \psi_2^- \le |u|$ a.e. in Ω.
Since $u \in L^q(\Omega)$, by 1) and 2) we have $\psi_1^+, \psi_2^- \in L^q(\Omega)$.

By (G.3) and Prop.3.3 a), it results that $G(\cdot,\psi_1^+)$, $G(\cdot,\psi_2^-) \in L^1(\Omega)$. □

Now, let us study the properties of $f = f_0 + I_V$.
Since, we can assume $\Phi(x,0) = 0$ a.e. in Ω, by $(\Phi.1)-(\Phi.4)$, we deduce that $V = \left\{v \in H : \int_\Omega \Phi(x,v)dx = \rho\right\}$ $(\rho \neq 0)$, is a hypersurface of class C^2 in H.

REMARK 3.6 —

a) *Under the assumptions* $(\Phi.1)-(\Phi.2)$, *we have:*

$$\partial^- I_V(u) = \{\lambda\,\eta(u) : \lambda \in \mathbb{R}\} \quad \forall u \in V$$

where $\eta : H \longrightarrow H$ *verifies,*

$$<\eta(u), v> = \int_\Omega \phi(x,u)v dx \quad \forall v \in H$$

b) Let u belong to $V \cap K$. V and K are tangent at u if and only if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that: $\quad < \lambda\eta(u), v - u > \leq 0 \quad \forall v \in K$.

Let us state a result on the characterization of the condition of nontangency, recalling that other results can be found in [2], [3] and [12].

THEOREM 3.7 — *Let Φ a function verifying $(\Phi.1), (\Phi.2), (\Phi.4)$. Let V, K and u_K be as already defined (see sect.2).*
If $u \in K \cap V$, then the following statements are equivalents :

a) K and V are tangent at u ;

b) $u = u_K$ or

$$meas(\{x \in \Omega : \psi_1(x) < u(x) < 0\} \cup \{x \in \Omega : 0 < u(x) < \psi_2(x)\}) = 0.$$

By combining Prop.3.5 with Theo.1.5 of [12], we can state

THEOREM 3.8 — *If $D(f_0)$ and V are not tangent at any point, under the assumptions $(G.1), (G.2), (\Phi.1) - (\Phi.4)$ (and (Ψ) in the case $p > 2, q > \frac{2n}{n-2}$), then:*

a) f is lower semicontinuous and of class $C(P, Q)$ where P and Q are suitable continuous functions on $D(f) \times D(f)$ with the topology of H.

b) $\quad \partial^- f(u) = \partial^- f_0(u) + \partial^- I_V(u) \quad \forall u \in D(f)$.

c) If $u \in D(f)$ and $\alpha \in H$, then:
$\alpha \in \partial^- f(u)$ if and only if there exists $\lambda \in \mathbb{R}$ such that :

$$\int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx +$$

$$+\lambda \int_\Omega \phi(x,u)(v-u)dx \geq < \alpha, v-u > \quad \forall v \in D(f_0).$$

Finally, we can conclude this section giving the proof of the characterization theorem.

PROOF of THEO. 3.1 .
By a_4) of Prop.3.5, if (u, λ) is a solution of (P), then $-\lambda\eta(x,u) \in \partial^- f_0(u)$. Since $\partial^- f_0(u) \cup \partial^- I_V(u) \subseteq \partial^- f(u)$, then $0 \in \partial^- f(u)$.

Viceversa, if $f(u) < +\infty$ and $0 \in \partial^- f(u)$, by b) of Theo.3.8, we have:

$$\partial^- f(u) = \partial^- f_0(u) + \partial^- I_V(u) \quad \forall u \in D(f).$$

So, by Remark 3.6 a), there exist $\lambda \in \mathbb{R}$ and $\alpha \in \partial^- f_0(u)$ such that

$0 = \alpha + \lambda\, \eta(u)$. Then, by Prop.3.5 a_4), we have

$$\int_\Omega |Du|^{p-2} DuD(v-u)dx - \int_\Omega g(x,u)(v-u)dx \geq$$

$$\geq -\lambda \int_\Omega \phi(x,u)(v-u)dx \quad \forall v \in K_g,$$

i.e., (u, λ) is a solution of (P). □

4. TOPOLOGICAL RESULTS AND THE PROOF OF THE MAIN THEOREMS

In this section, we will assume that g and ϕ are odd (with respect to the variable t), $\psi_2 = -\psi_1 = \psi$ and that the assumptions in Theo.3.8 hold.

We want to prove that the functional f, defined in the previous section, has infinitely many critical points on H, that is, by Theo.3.1, there exist infinitely many solutions of the problem (P).

To this aim, let us state a theorem linking the category of a space with the number of critical points of a functional defined on it. Let us recall two definitions.

DEFINITION 4.1 — *Let X be a real Hilbert space, $J : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semicontinuous functional.*
We set $d^*(u,v) = \|u-v\|_X + |J(u) - J(v)| \quad \forall u, v \in D(J)$ and we denote by $D(J)^*$ the metric space $(D(J), d^*)$.
Let $A \subset D(J)$ be a symmetric subset with respect to the origin. Then A is said to be Z_2-categorical in $D(J)^*$, if there exist $u \in D(J)$ and a d^*-continuous deformation $F : A \times [0,1] \to D(J)$ such that

$$F(A,1) \subset \{u, -u\} \text{ and } F(-v,t) = -F(v,t) \quad \forall t \in [0,1], \forall v \in A$$

DEFINITION 4.2 — *Let A be a symmetric, closed subset of $D(J)^*$. We denote by Z_2-cat$(A; D(J)^*)$ the least integer n such that A can be covered by n closed symmetric subsets of $D(J)^*$, each of which is Z_2-categorical in $D(J)^*$. If no such integer n exists, we put Z_2-cat$(A; D(J)^*) = +\infty$. We set, also, Z_2-cat$(\emptyset; D(J)^*) = 0$, Z_2-cat$(D(J)^*) = Z_2$-cat$(D(J)^*; D(J)^*)$.*

THEOREM 4.3 — *(see [8]) Let X be a real Hilbert space, $J : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semicontinuous functional with a φ-monotone subdifferential of order two.*

Let us suppose that:

a) J is lower bounded;

b) $(P\text{-}S)_c$: $\forall c \in J(D(J))$, $\forall (u_h)_h \subset D(\partial^- J)$ with

$$\lim_h \|grad^- J(u_h)\|_X = 0 \text{ and } \sup_h J(u_h)' < c$$

$(u_h)_h$ has a subsequence converging (with respect to d^*) in X;

c) Z_2-$cat(D(J)^*) = +\infty$.

Then, there exists a sequence $(c_h)_h$ of distinct lower critical values of J with $\lim_{h\to\infty} c_h = \sup_{D(J)} J$.

We will apply this theorem to the functional f and the space H.

Now, let us verify the hypotheses of Theo.4.3.

PROPOSITION 4.4 — *f verifies the hypothesis a) in Theo. 4.3.*

PROOF. Let us take $u \in D(f_0) \cap V$,then, by $(G.1)$, we have

$$f_0(u) = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega G(x,u)dx \geq$$

$$\geq \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega a_0(x)dx - b_0\|u\|^q_{L^q(\Omega)}. \tag{4.4.1}$$

From this, since $u \in K$, by (Ψ) we get

$$f_0(u) \geq - \int_\Omega a_0(x)dx - c\|\widetilde{\psi}\|^q_{L^r(\Omega)}$$

where $\widetilde{\psi} = \max(|\psi_1|, |\psi_2|)$ and c depends only on n and Ω.

So a) is proved. □

PROPOSITION 4.5 — *f verifies $(P\text{-}S)_c$ property.*

PROOF. Let $(v_m)_m$ a sequence in $D(\partial^- f)$ such that $f(v_m) < c$.
By (4.4.1) and (Ψ), we get that $(v_m)_m$ is a bounded sequence in $W_0^{1,p}(\Omega)$.
Let us still denote by $(v_m)_m$ a subsequence weakly converging to v in $W_0^{1,p}(\Omega)$.
By Rellich's theorem, $v_m \to v$ in $L^q(\Omega)$.
Let us set $\alpha_m = grad^- f(v_m)$. Since K_g and V are not tangent at v_m we can apply Remark 3.6 and b) of Theo.3.8. Thus, there exist two sequences

$(\lambda_m)_m \subset \mathbb{R}$ and $(\gamma_m)_m \subset \partial^- f_0(v_m)$ such that:

$$\alpha_m = \gamma_m + \lambda_m \eta(v_m).$$

Let us recall that $\eta(v_m) \to \eta(v)$, since η is a compact map.

To prove the thesis, it is enough to prove that:

i) there exists a subsequence $(v_{m_j})_j$ converging to v in H.

ii) $(f(v_{m_j}))_j$ converges to $f(v)$.

Let us start with i).

In the case $\frac{2n}{n+2} < p < 2$, the statement follows by compactness of $i : W_0^{1,p}(\Omega) \hookrightarrow H$.

Now, let us suppose $p \geq 2$.

There exists a subsequence $(\gamma_{m_j})_j$ strongly convergent in H.

Indeed, by [2], since K_g and V are not tangent in v, there exist w_1 and w_2 belonging to K_g such that:

$$0 << \eta(v), w_1 - v >= \int_\Omega \phi(x,v)(w_1 - v) =$$

$$= \lim_{m\to\infty} \int_\Omega \phi(x, v_m)(w_1 - v_m) \tag{4.5.1}$$

and

$$0 >< \eta(v), w_2 - v >= \int_\Omega \phi(x,v)(w_2 - v) =$$

$$= \lim_{m\to\infty} \int_\Omega \phi(x, v_m)(w_2 - v_m). \tag{4.5.2}$$

By (4.5.1) and (4.5.2) there exist $\epsilon > 0$ and $m_0 \in N$ such that

$$< \eta(v_m), w_2 - v_m >< -\epsilon < \epsilon << \eta(v_m), w_1 - v_m > \quad \forall m \geq m_0.$$

Moreover, since $\gamma_m \in \partial^- f_0(v_m)$, by b) of Prop.3.3 and by Prop.3.5, we have:

$$f_0(w_1) \geq f_0(v_m) + < \gamma_m, w_1 - v_m > +$$

$$-c\left\{\|a_1\|_{L^r(\Omega)} + \|w_1\|^{q-2}_{W_0^{1,p}(\Omega)} + \|v_m\|^{q-2}_{W_0^{1,p}(\Omega)}\right\} \|w_1 - v_m\|^2_{W_0^{1,p}(\Omega)}.$$

Then, $< \gamma_m, w_1 - v_m >$ is upper bounded.

Since $\alpha_m \to 0$ and

$$< \alpha_m, w_1 - v_m >=< \gamma_m, w_1 - v_m > + \lambda_m < \eta(v_m), w_1 - v_m >,$$

we obtain that $(\lambda_m)_m$ is lower bounded.
Analogously, we obtain that $< \gamma_m, w_2 - v_m >$ is upper bounded.
Thus, $(\lambda_m)_m$ is bounded and there exists a subsequence $(\lambda_{m_j})_j$ converging in $\mathbb{R}$. So, the sequence $(\gamma_{m_j})_j$, with $\gamma_{m_j} = \alpha_{m_j} - \lambda_{m_j}\eta(v_{m_j})$, converges in H.

Now, to prove i) let us suppose that $\gamma_{m_j} \to \gamma$ in H.
Since $\gamma_{m_j} \in \partial^- f_0(v_{m_j})$, by a_4) of Prop.3.5, we have:

$$< \gamma_{m_j}, v - v_{m_j} > \leq \int_\Omega |Dv_{m_j}|^{p-2} Dv_{m_j} D(v - v_{m_j})dx - \int_\Omega g(x, v_{m_j})(v - v_{m_j})dx. \tag{4.5.3}$$

Moreover, since $\gamma_{m_j} \to \gamma$ in H and $v_{m_j} \to v$ weakly in $W_0^{1,p}(\Omega)$, by a_2) of Prop. 3.5, $\gamma \in \partial^- f_0(v)$ and, again by a_4) of Prop.3.5, we have

$$< \gamma, v_{m_j} - v > \leq \int_\Omega |Dv|^{p-2} Dv D(v_{m_j} - v)dx - \int_\Omega g(x, v)(v_{m_j} - v)dx. \tag{4.5.4}$$

By adding (4.5.3) and (4.5.4), we have:

$$< \gamma_{m_j} - \gamma, v_{m_j} - v > + \int_\Omega [g(x, v_{m_j}) - g(x, v)] (v_{m_j} - v)dx \geq$$

$$\geq \int_\Omega \left[|Dv_{m_j}|^{p-2} Dv_{m_j} - |Dv|^{p-2} Dv\right] D(v_{m_j} - v)dx. \tag{4.5.5}$$

Let us point out that the first member in (4.5.5) tends to 0 for $j \to \infty$. Since $(v_{m_j})_j$ converges weakly to v in $W_0^{1,p}(\Omega)$, to conclude the proof of i), it is sufficient to prove that :

$$\int_\Omega \left[|Dv_{m_j}|^{p-2} Dv_{m_j} - |Dv|^{p-2} Dv\right] D(v_{m_j} - v)dx \geq$$

$$\geq \left| \|v_{m_j}\|_{W_0^{1,p}(\Omega)} - \|v\|_{W_0^{1,p}(\Omega)} \right|^p. \tag{4.5.6}$$

By Cauchy-Schwarz's inequality on $\mathbb{R}^n$ and by Hölder's inequality, we have :

$$\int_\Omega \left[|Dv_{m_j}|^{p-2} Dv_{m_j} - |Dv|^{p-2} Dv\right] D(v_{m_j} - v)dx \geq$$

$$\geq \| \, |Dv_{m_j}| \, \|_{L^p(\Omega)}^p - \| \, |Dv_{m_j}|^{p-1} \, \|_{L^{p'}(\Omega)} \| \, |Dv| \, \|_{L^p(\Omega)} +$$

$$- \| \, |Dv|^{p-1} \, \|_{L^{p'}(\Omega)} \| \, |Dv_{m_j}| \, \|_{L^p(\Omega)} + \| \, |Dv| \, \|_{L^p}^p =$$

$$= \left(\|v_{m_j}\|_{W_0^{1,p}(\Omega)}^{p-1} - \|v\|_{W_0^{1,p}(\Omega)}^{p-1} \right) \left(\|v_{m_j}\|_{W_0^{1,p}(\Omega)} - \|v\|_{W_0^{1,p}(\Omega)} \right). \tag{4.5.7}$$

Then, (4.5.6) follows by (4.5.7) and by the following real inequality :

$$\left(|t|^{p-2}t - |s|^{p-2}s\right)(t - s) \geq |t - s|^p, \quad \forall s, t \in \mathbb{R}, p \geq 2. \tag{4.5.8}$$

Because of homogeneity, to prove (4.5.8) we can consider the case $s = 1$, $|t| \leq 1$ and it is sufficient to study the real function:

$$h(t) = \frac{(|t|^{p-2}t - 1)(t - 1)}{|t-1|^p}, -1 \leq t < 1.$$

Now let us prove ii). By a) of Theo.3.8, we have:

$$f(v) \geq f(v_{m_j}) + < \alpha_{m_j}, v - v_{m_j} > - [P(v, v_{m_j})\|\alpha_{m_j}\| + Q(v, v_{m_j})] \|v_{m_j} - v\|^2$$

where P and Q are suitable continuous real functions on $D(f) \times D(f)$. Thus, it results $\limsup_{j\to\infty} f(v_{m_j}) \leq f(v)$. By the lower semicontinuity of f, the thesis is completely proved. □

Before dealing with the Z_2-category of $K_g \cap V$, let us state the following lemma.

LEMMA 4.6 — *Let Ω be a bounded open subset of $\mathbb{R}^n$. Then $\forall m \in \mathbb{N}$ there exist $w_1, w_2, ... w_m \in W_0^{1,p}(\Omega)$ $(p \geq 1)$ such that if $\sum_{j=1}^m c_j w_j = 0$ a.e. in a subset of Ω with strictly positive measure, with $c_1, c_2, ... c_m \in \mathbb{R}$, then $c_1, c_2, ... c_m = 0$.*

PROOF. - Let $R > 0$ be such that $\overline{\Omega} \subset B(0, R)$ and let us fix $m \in \mathbb{N}$. Let us consider $u_1, u_2, ... u_m$ such that :

$$\begin{cases} u_j \in W_0^{1,2}(B(0,R)) \\ -\Delta u_j = \lambda_j u_j \text{ on } B(0,R) \end{cases}$$

Then $u_j \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \quad \forall p \geq 1$.
Now, let us take $\theta \in C^\infty(\mathbb{R}^n)$ such that: $\Omega = \{x \in \mathbb{R}^n : \theta(x) > 0\}$, and let us set $w_j = \theta u_j$.
Then $w_j \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $w_j = 0$ on $\partial\Omega$. So, $w_j \in W_0^{1,p}(\Omega)$.
Now, by definition of w_j, if $\sum_{j=1}^m c_j w_j(x) = 0$ a.e. in $E \subset \Omega$ with $\text{meas}(E) > 0$, then $\sum_{j=1}^m c_j u_j(x) = 0$ a.e. in $E \subset \Omega \subset B(0, R)$.
Moreover every u_j is analytic in $B(0, R)$, thus we have
$c_1 = c_2 ... = c_m = 0$. □

PROPOSITION 4.7 — f *verifies the assumption c) in Theo. 4.3, i.e.* $Z_2\text{-}cat(D(f)^*) = +\infty$.

PROOF. - Let W be the linear manifold generated by the functions $w_1, ... w_m \in W_0^{1,p}(\Omega)$ of Lemma 4.6. Let us set $S' = V \cap W$. Let us point out that $S' \subseteq W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Since, by hypotheses $(\Phi_1), (\Phi_2), (\Phi_4)$, V is radially homeomorphic to $\{v \in W_0^{1,p}(\Omega) : \|v\| = 1\}$, we have:

$$m \leq Z_2\text{-}cat(S', W_0^{1,p}(\Omega)\setminus\{0\}).$$

Now, let us define the map

$$F_1 : S' \times [0,1] \to W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\setminus\{0\}$$

in such way:

$$F_1(v,t) = (1-t)(v-w)^+ + (v \vee (-w)) \wedge w - (1-t)(v+w)^-$$

where $w \in K_g$ is such that $w \geq 0$ and

$$\rho < \int_\Omega \Phi(x,w)\, dx < \sup_{v \in K_g} \int_\Omega \Phi(x,v)\, dx.$$

It is easy to see that $F_1(v,0) = v$, $F_1(v,1) \in K_g$ and

$$\int_\Omega \Phi(x, F_1(v,1))\, dx \leq \int_\Omega \Phi(x,v)\, dx = \rho.$$

Also, let us consider the map $F_2 : (K_g\setminus\{0\}) \times [0,1] \to K_g\setminus\{0\}$ defined by

$$F_2(v,t) = \left(\left(\frac{1}{1-t}v\right) \wedge w\right) \vee (-w).$$

By lemma 4.6, we have:

$$\lim_{t \to 1^-} \int_\Omega \Phi(x, F_2(v,t))\, dx = \int_\Omega \Phi(x,w)\, dx > \rho \quad \forall v \in F_1(S',1).$$

Since $F_2(S',1)$ is compact, there exists $\bar{t} \in]0,1[$ such that

$$\int_\Omega \Phi(x, F_2(v,\bar{t}))\, dx \geq \rho \quad \forall v \in F_1(S',1),$$

By hypothesis $(\Phi_1), (\Phi_2), (\Phi_4)$ we can define the map

$$P : W_0^{1,p}(\Omega)\setminus\{0\} \to W_0^{1,p}(\Omega)\setminus\{0\}$$

in the following way:

$$P(v) = \tau(v)v$$

where $\tau : W_0^{1,p}(\Omega)\backslash\{0\} \to \mathbb{R}^+$ is continuous and $\int_\Omega \Phi(x, P(v))\, dx = \rho \;\forall v \neq 0$.
Let us point out that $\forall v \neq 0$ with $\int_\Omega \Phi(x,v)\, dx \geq \rho$, we have $\tau(v) \leq 1$.
Finally, we are able to consider the deformation map

$$F : S' \times [0, 1+\bar{t}] \to W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\backslash\{0\}$$

defined as

$$F(v,t) = \begin{cases} F_1(v,t) & v \in S',\, t \in [0,1] \\ P(F_2(F_1(v,1), t-1)) & v \in S',\, t \in [1, 1+\bar{t}]. \end{cases}$$

We have $F(v, 0) = v$, and $F(v, 1+\bar{t}) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap V$.
Moreover, F is odd in the first variable: $F(v,t) = -F(-v,t)$.
Let us observe that F is continuous when S' and the codomain are equipped with the $W_0^{1,p}(\Omega)$-topology.
Thus

$$m \leq Z_2\text{-}cat(S', W_0^{1,p}(\Omega)\backslash\{0\}) \leq Z_2\text{-}cat(F(S', 1+\bar{t}), W_0^{1,p}(\Omega)\backslash\{0\}) \leq$$

$$\leq Z_2\text{-}cat(F(S', 1+\bar{t}), D(f)).$$

It is easy to see that F is also continuous when the topology considered is that one induced by the metric d^*.
Then

$$m \leq Z_2\text{-}cat(F(S', 1+\bar{t}), D(f)) \leq Z_2\text{-}cat(F(S', 1+\bar{t}), D(f)^*) \leq$$

$$\leq Z_2\text{-}cat(D(f)^*).$$

□

Finally, we can prove the main theorems.

PROOF of THEO. 2.2.

By Theo 3.1, it is enough to prove that there exists $u \in K_g \cap V$ such that $0 \in \partial^- f(u)$. Let us observe that since f is lower bounded and, by Prop.4.5 satisfies $(\text{P-S})_c$, there exists $u \in K_g \cap V$ such that u is a minimal point for f and thus $0 \in \partial^- f(u)$. □

PROOF of THEO. 2.3.

By Prop. 4.4, 4.5 and 4.7, we can use Theo. 4.3 to obtain that there exists a sequence $(u_k)_k \in K_g \cap V$ such that u_k and $-u_k$ are lower critical points of f.

Then, by Theo. 3.1, there exists a real sequence $(\lambda_k)_k$ such that (u_k, λ_k) and $(-u_k, \lambda_k)$ are solutions of the problem (P).

Thus, it remains to prove that

$$\inf\{\lambda_k : k \in N\} = -\infty. \tag{2.3.1}$$

To prove this, let us observe that, if (u, λ) is a solution of (P), then:

$$\int_\Omega |Du|^p dx - \int_\Omega g(x,u)u dx + \lambda \int_\Omega \phi(x,u)u dx \leq 0.$$

Now, using this inequality and the assumption (G.2), by definition of f, it easy to see that (2.3.1) follows by

$$\sup\{f(v) : v \in K_g \cap V\} = +\infty. \tag{2.3.2}$$

To prove (2.3.2), let us take $w \in K_g \cap V$ such that:

$$w \geq 0, \int_\Omega \Phi(x,w)dx > \rho \text{ and } \int_\Omega \Phi(x,tw)dx < \rho,$$

for some $t \in (0,1)$ and let $w' \in L^\infty(\Omega) \backslash W_0^{1,p}(\Omega)$ with $tw \leq w' \leq w$.

Let us consider a sequence $(w_j)_j \subset W_0^{1,p}(\Omega)$ with $tw \leq w_j \leq w$ and $w_j \to w'$ *a.e.* in Ω. Since, $\lim_{j\to\infty} \|w_j\|_{W_0^{1,p}(\Omega)} = +\infty$, for some sequence $t_j \in [0,1]$ with $t_j w_j \in K_g \cap V$ we get

$$\lim_{j\to\infty} \|t_j w_j\|_{W_0^{1,p}(\Omega)} = +\infty.$$

From this, (2.3.2) follows. □

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Annamaria Canino, *Dipartimento di Matematica, Università della Calabria, Arcavacata di Rende (CS), Italy.*

Umile Perri, *Facoltà d'Ingegneria, Università di Reggio Calabria, Via E. Cuzzocrea, Reggio Calabria, (Italy).*

PERTURBAZIONI SINGOLARI PER EQUAZIONI QUASI ELLITTICHE IN L^P

Nota di Ornella FIODO e Salvatore GIUGA (a Napoli) (*) (**)
Presentata dal socio GUIDO TROMBETTI
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Riassunto — *In questo lavoro si considera nel semispazio e nell'ambito L^pun problema di perturbazioni singolari per equazioni quasi ellittiche con condizioni al bordo di Dirichlet e si studiano le proprietà di convergenza delle soluzioni.*

Abstract — *In this paper we consider in the half-space and in L^p a problem of quasi elliptic singular perturbations with Dirichlet boundary conditions and we study the convergence of the solutions.*

In [8] O. FIODO (cfr. anche [7]), nella linea di D. HUET (cfr. [10]), ha considerato una classe di perturbazioni singolari per operatori quasi ellittici di q-ordine m, a coefficienti variabili, del tipo:

$$\begin{cases} \epsilon^m A(x,D)u + u = f_\epsilon & \text{in } R^n_+ \quad (\epsilon > 0) \\ B_j^{(\epsilon)}(x',D)u(x',0) = g_j^{(\epsilon)} & j=1,\cdots,\nu, \end{cases}$$

ed ha dimostrato che, se $f_\epsilon \to f$ per $\epsilon \to 0$ in $L^2(R^n_+)$ allora $u_\epsilon \to f$ in $L^2(R^n_+)$; se inoltre avviene, per qualche $s>0$, che $f_\epsilon \to f$ in $H^{s,2}_{loc}(R^n_+)$, allora $u_\epsilon \to f$ in $H^{s,2}_{loc}(R^n_+)$.

In questo lavoro dimostriamo che, relativamente al problema di Dirichlet, tali risultati continuano a sussistere anche in ambiente L^p con $p \in]1,+\infty[$. Come in [8], il problema è inquadrato nell'ambito degli operatori dipendenti da parametro studiati, nel caso ellittico, da AGRANOVICH e VISHIK in ambiente L^2 (cfr.[1]), da A. ALVINO e G. TROMBETTI in ambiente L^p (cfr.[2]) e da E. GIARRUSSO (cfr. [9]) nel caso di operatori quasi ellittici.
Nello studio della convergenza della soluzione u_ϵ in $L^p(R^n_+)$ abbiamo utilizzato i procedimenti di [8]; invece, in quello della convergenza di u_ϵ in $H^{s,p}_{loc}(R^n_+)$, siamo ricorsi ad un artificio di tipo pseudodifferenziale (cfr. Prop. 3.3), il quale, tra l'altro, ci ha consentito di ottenere risultati che sono più generali di quelli di [8] anche nel caso $p=2$; ciò nel senso di una minore regolarità richiesta ai coefficienti della parte principale dell'operatore $A(x,D)$.

Il lavoro si articola in 4 paragrafi. Nel §1 si introducono gli spazi $H^{s,p}$ e $B^{s,p}$ e si richiamano alcune loro proprietà; nel §2 si studia il problema di perturbazioni singolari in $L^p(R^n_+)$; nel §3 si stabiliscono alcune formule di maggiorazione per gli operatori di convoluzione relativi a derivate di ordine non intero; infine nel §4 si dimostra il teorema sulla convergenza della soluzione u_ϵ negli spazi $H^{s,p}_{loc}(R^n_+)$.

§1 *Preliminari* — Siano: R^n, lo spazio euclideo a n dimensioni di punto $x=(x_1,\cdots,x_n)=(x',x_n)$; R_n il duale di R^n, di punto $\xi=(\xi_1,\cdots,\xi_n)=(\xi',\xi_n)$; R^n_+ il semispazio $\{x\in R^n\colon\ x_n>0\}$.

Se α è un multiindice di N^n , poniamo:

$$D^\alpha=D^{\alpha_1}_{x_1}\cdots D^{\alpha_n}_{x_n} \quad , \quad D_{x_j}=i^{-1}\partial_{x_j} \qquad (i=\sqrt{-1})$$

$$\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \qquad\qquad <\xi,x>=\xi_1 x_1+\cdots+\xi_n x_n;$$

Indicheremo con F e F^{-1} rispettivamente la trasformata e l'antitrasformata di Fourier :

$$F\varphi(\xi)=\int_{R^n}\exp(-i<x,\xi>)\varphi(x)dx \qquad \forall\varphi\in S(R^n),$$

$$F^{-1}\varphi(x)=(2\pi)^{-n}\int_{R_n}\exp(i<x,\xi>)\varphi(\xi)d\xi \qquad \forall\varphi\in S(R_n).$$

dove $S(R^n)$ e $S(R_n)$ sono gli spazi delle funzioni a decrescenza rapida, rispettivamente in x e in ξ.

—*Gli spazi* $H^{s,p}(R^n)$ *e* $H^{s,p}(\Omega)$ — Sia $(m_1,\cdots,m_n)$ una n-pla di interi positivi; poniamo:

$$m=\max_j m_j, \quad q_j=\frac{m}{m_j} \quad \forall j\in\{1,\cdots,n\} , \qquad <\alpha,q>=\sum_{j=1}^n\alpha_j q_j.$$

e introdotta in R_n, la funzione :

$$\Lambda(\xi)=\left(1+\sum_{j=1}^n\xi_j^{2m_j}\right)^{1/2m},$$

per ogni $p>1$, $s\geq 0$, denotiamo con $H^{s,p}(R^n)$ lo spazio delle distribuzioni temperate tali che $F^{-1}\Lambda^s(\xi)Fu\in L^p(R^n)$, munito della norma:

$$\|u\|_{s,p}=\left\|F^{-1}\Lambda^s(\xi)Fu\right\|_{0,p}.$$

È evidente che $S(R^n)$ è denso in $H^{s,p}(R^n)$; inoltre posto:

$$s_j = \frac{s}{q_j} \qquad \forall j \in \{1,\cdots,n\}$$

si dimostra che:

1.1 — *Per ogni* $s \geq 0$ *lo spazio* $H^{s,p}(R^n)$ *risulta isomorfo algebricamente e topologicamente allo spazio delle* $u \in L^p(R^n)$ *tali che:*

$$F^{-1}(1+\xi_j^2)^{s_j/2} Fu \in L^p(R^n) \qquad \forall j \in \{1,\cdots,n\},$$

munito della norma: $\sum_{j=0}^{n} \left\| F^{-1}(1+\xi_j^2)^{s_j/2} Fu \right\|_{0,p}$.

Sia ora Ω un aperto di R^n. Per ogni $p \in]1,+\infty[$, $s \geq 0$, denotiamo con $H^{s,p}(\Omega)$ lo spazio delle distribuzioni u su Ω che sono restrizioni a Ω di elementi U di $H^{s,p}(R^n)$; esso è munito della norma:

$$\|u,\Omega\|_{s,p} = \inf_{U/\Omega = u} \|U\|_{s,p} \qquad , U \in H^{s,p}(R^n).$$

Per uno studio dettagliato di tali spazi rimandiamo a [17] (caso isotropo) e a [12] (caso anisotropo); qui ci limitiamo a richiamare le proprietà che verranno utilizzate in questa nota.

1.2 — $C_0^\infty(\overline{\Omega})$ *è denso in* $H^{s,p}(\Omega)$ *per* $s>0$.

1.3 — *Se* $0<s_1<s_2$ *risulta* $H^{s_2,p}(\Omega) \subset H^{s_1,p}(\Omega)$, *con immersione continua.*

1.4 — *Se* $u \in H^{s,p}(\Omega)$, *allora per ogni multiindice* α *tale che* $<\alpha,q> \leq s$, *l'operatore di derivazione* D^α *è continuo:*

$$H^{s,p}(\Omega) \rightarrow H^{s-<\alpha,q>,p}(\Omega)$$

1.5 — *Se* $\Omega'' \subseteq \Omega'$, *abbiamo:* $H^{s,p}(\Omega') \subseteq H^{s,p}(\Omega'')$, *con immersione continua.*

—Gli spazi $B^{s,p}(R^{n-1})$*—* Per $n\geq 2$, denotato con $\Delta_{j,t}$, $t\in R$, l'operatore:

$$\Delta_{j,t}f(x) = f(x_1,\cdots,x_j+t,\cdots,x_n) - f(x_1,\cdots,x_n),$$

indichiamo con $B^{s,p}(R^{n-1})$, $s>0$ $p>1$, (cfr. [5], [13] e [16]) lo spazio delle distribuzioni $u\in L^p(R^{n-1})$ tali che, $\forall j\in\{1,\cdots,n-1\}$:

$$I_{s_j,p}(u)=\int_0^{+\infty} t^{-(1+p(s_j-s_j^*))}dt\int_{R^{n-1}} \left|\Delta_{j,t}^{k_j}D_{x_j}^{s_j^*}u\right|^p dx'<+\infty$$

dove s_j^* è il più grande intero minore di s_j, e k_j è uguale: a 1 se $s_j\notin N$, a 2 se $s_j\in N$; poniamo inoltre:

$$\|u\|_{B^{s,p}(R^{n-1})} = [\![u]\!]_{s,p} = \|u\|_{L^p(R^{n-1})}+\sum_{j=0}^{n-1}\left\{I_{s_j,p}(u)\right\}^{1/p}.$$

Se, per ogni $u\in C_0^\infty(\overline{R}_+^n)$, indichiamo con $\gamma_0 u$ la traccia di u sull'iperpiano $x_n=0$ abbiamo:

1.6—*Siano* $p>1$ *e* $s_n>1/p$. *Se* $k-1$ *è il massimo intero non negativo minore di* s_n-1/p, *l'applicazione*:

$$u\in C_0^\infty(\overline{R}_+^n) \to (\gamma_0 u,\gamma_0 D_{x_n}u,\cdots,\gamma_0 D_{x_n}^{k-1}u)\in\prod_{h=0}^{k-1}C_0^\infty(R^{n-1}),$$

si prolunga in un operatore lineare e continuo:

$$H^{s,p}(R_+^n)\to \prod_{h=0}^{k-1}B^{s-(h+\frac{1}{p})q_n,p}(R^{n-1}).$$

Inoltre se:

$$(g_0,\cdots,g_{k-1})\in \prod_{h=0}^{k-1}B^{s-(h+\frac{1}{p})q_n,p}(R^{n-1}),$$

$\forall\epsilon>0$ *esiste almeno una distribuzione* $w_\epsilon\in H^{s,p}(R^n_+)$ *tale che* $\gamma_0 D^h_{x_n} w_\epsilon = g_h$ $\forall\, h=0,1,\cdots,k\text{-}1$ *e risulta:*

$$\text{(1.1)}\qquad \|w_\epsilon,\, R^n_+\|_{0,p} + \epsilon^s \|w_\epsilon,\, R^n_+\|_{s,p}$$

$$\leq C \sum_{h=0}^{k-1} \left\{ \epsilon^{(h+1/p)q_n} [\![g_h]\!]_{0,p} + \epsilon^s [\![g_h]\!]_{s-(h+\frac{1}{p})q_n,p} \right\}$$

con C costante indipendente da ϵ *e* $(g_0,\cdots,g_{k-1})$.

— *Gli spazi* M_s — Per ogni $j\in\{1,\cdots,n\}$, denotiamo con M_{s_j} lo spazio delle funzioni $\psi\in L^\infty(R^n)$ per le quali risulta:

$$\|\psi\|_{M_{s_j}} = \sum_{k=0}^{[s_j]} \left(\|D^k_{x_j}\psi\|_\infty + \int_R |t|^{-(\sigma_j+1)} \|\Delta_{j,t} D^k_{x_j}\psi\|_\infty dt \right) < +\infty ,$$

dove $[s_j]$ è il più grande intero non superiore a s_j e $\sigma_j = s_j - [s_j]$; poniamo inoltre (cfr. [3] e [9]):

$$M_s = \bigcap_{j=1}^{n} M_{s_j}$$

con la norma:

$$\|\psi\|_{M_s} = \sum_{j=1}^{n} \|\psi\|_{M_{s_j}}.$$

È utile, per il seguito, osservare che $C^\infty_0(R^n)\subseteq M_s$, $\forall s\geq 0$. Sussiste la proposizione (cfr [9]):

1.7 — *Per ogni* $s\geq 0$, $\psi\in M_s$, *e* $u\in H^{s,p}(\Omega)$ *si ha:*

$$\|\psi u,\, \Omega\|_{s,p} \leq C \|\psi\|_{M_s} \|u,\, \Omega\|_{s,p}$$

con C costante indipendente da u.

§2 — *Perturbazioni singolari* — Sia $n \geq 2$. Consideriamo in R^n_+ l'operatore:

$$A(x,D) = \sum_{<\alpha,q> \leq m} a_\alpha(x) D^\alpha,$$

e la famiglia di parametro $\epsilon > 0$:

$$\mathcal{A}_\epsilon(x,D) = \epsilon^m A(x,D) + I.$$

Posto:

$$A_0(x,D) = \sum_{<\alpha,q> = m} a_\alpha(x) D^\alpha,$$

supponiamo verificate le seguenti ipotesi:

(1) *i coefficienti* $a_\alpha(x)$ *di* $A(x,D)$ *sono di classe* $M_{s+m-<\alpha,q>}$, *con* $s \geq 0$, *fissato;*

(2) *i coefficienti di* $A_0(x,D)$ *sono continui e convergenti all'*∞;

(3) *per ogni* $x \in \overline{R}^n_+$ *risulta:*

$$A_0(x,\xi) \neq 0 \quad \forall \xi \in R_n - \{0\}, \qquad A_0(x,\xi) + 1 \neq 0 \quad \forall \xi \in R_n;$$

(4) *per ogni* $\xi' \in R_{n-1} - \{0\}$ *è costante il numero* ν *delle radici con parte immaginaria positiva dell'equazione in* τ: $A_0(0,\xi',\tau) = 0$ $(\tau \in \mathbb{C})$.

Dall'ultima ipotesi segue che $\forall x' \in R^{n-1} \cup \{\infty\}$ e $\forall \xi' \neq 0$ (risp. $\forall \xi'$), le radici con parte immaginaria positiva dell'equazione:

$$A_0((x',0),\xi',\tau) = 0 \qquad (\textit{risp. } A_0((x',0),\xi',\tau) + 1 = 0)$$

sono ancora ν.

Ciò posto, consideriamo il seguente problema di perturbazioni singolari:

$$(2.1)\qquad \begin{cases} \mathcal{A}_\epsilon(x,D)u_\epsilon = f_\epsilon \\ \gamma_0(D^h_{x_n}u_\epsilon) = g^{(\epsilon)}_h, \quad h=0,\cdots,\nu-1 \end{cases}$$

dove supponiamo che, $\forall\epsilon>0$, risulti:

$$(2.2)\qquad f_\epsilon \in H^{s,p}(R^n_+)$$

$$(2.3)\qquad g^{(\epsilon)}_h \in B^{s+m-(h+1/p)q_n,p}(R^{n-1}), \quad \forall h\in\{0,1,\cdots,\nu-1\}.$$

Tale problema si inquadra nell'ambito dei problemi dipendenti da un parametro studiati in [9], assumendo $\lambda=1/\epsilon$ con $\epsilon\in]0,\infty[$. Dai risultati di tale nota si trae che:

2.1 — *Nelle ipotesi* (1), (2), (3), *per ogni* $s\geq 0$ *esiste un numero positivo* ϵ_0 *tale che,* $\forall\epsilon<\epsilon_0$, *il problema* (2.1) *ammette un'unica soluzione* $u_\epsilon\in H^{s+m,p}(R^n_+)$; *per essa si ha:*

$$(2.4)\qquad \|u_\epsilon\|^+_{0,p} + \epsilon^{s+m}\|u_\epsilon\|^+_{s+m,p} \leq C\Big\{\|f_\epsilon\|^+_{0,p} + \epsilon^s\|f_\epsilon\|^+_{s,p} + \sum_{h=0}^{\nu-1}\Big[\epsilon^{(h+1/p)q_n}[\![g^{(\epsilon)}_h]\!]_{0,p} + \epsilon^{s+m}[\![g^{(\epsilon)}_h]\!]_{s+m-q_n(h+1/p),p}\Big]\Big\}$$

dove C *è indipendente da* ϵ, *e dipende esclusivamente da* s *e dai coefficienti dell'operatore* $A(x,D)$.

Dimostriamo ora il seguente:

TEOREMA 2.2 — *Nelle ipotesi* (1), (2), (3), *e se inoltre:*

i) $$\lim_{\epsilon\to 0} \|f_\epsilon - f\|_{0,p} = 0$$

ii) $$\lim_{\epsilon\to 0}\sum_{h=0}^{\nu-1}\Big[\epsilon^{(h+1/p)q_n}[\![g^{(\epsilon)}_h]\!]_{0,p} + \epsilon^m[\![g^{(\epsilon)}_h]\!]_{m-q_n(h+1/p),p}\Big] = 0,$$

risulta anche:

(2.5)
$$\lim_{\epsilon \to 0} \|u_\epsilon - f\|_{0,p} = 0$$

(2.6)
$$\lim_{\epsilon \to 0} \epsilon^m \|u_\epsilon\|_{m,p} = 0$$

Dim. $\forall \epsilon < \epsilon_0$, sia w_ϵ un rilevamento di $(g_0^{(\epsilon)}, \cdots, g_{\nu-1}^{(\epsilon)})$ in $H^{m,p}(R_+^n)$ verificante la (1.1) (osserviamo in proposito che per l'ipotesi (3) risulta $\nu \leq m_n$ e quindi $\nu - 1 < m_n - 1/p$). Evidentemente dall'ipotesi *ii)* discende che:

$$\lim_{\epsilon \to 0} \|w_\epsilon\|_{0,p} = 0$$

posto allora $v_\epsilon = u_\epsilon - w_\epsilon$, la (2.5) è equivalente a:

(2.7)
$$\lim_{\epsilon \to 0} \|v_\epsilon - f\|_{0,p} = 0;$$

d'altra parte v_ϵ è soluzione del problema:

(2.8)
$$\begin{cases} \mathcal{A}_\epsilon(x,D) v_\epsilon = f_\epsilon \\ \gamma_0(D_{x_n}^h v_\epsilon) = 0, \quad h = 0, \cdots, \nu - 1 \end{cases}$$

per cui, non essendo restrittivo supporre $f \in C_0^\infty(R_+^n)$, $v_\epsilon - f$ è soluzione del problema:

(2.9)
$$\begin{cases} \mathcal{A}_\epsilon(x,D)(v_\epsilon - f) = f_\epsilon - f - \epsilon^m A(x,D) f \\ \gamma_0(D_{x_n}^h (v_\epsilon - f)) = 0, \quad h = 0, \cdots, \nu - 1 \end{cases}$$

dalla proposizione 2.1 con $s = 0$ traiamo allora che:

(2.10)
$$\|v_\epsilon - f\|_{0,p} \leq C(\|f_\epsilon - f\|_{0,p} + \epsilon^m \|A(x,D) f\|_{0,p}),$$

con C indipendente da ϵ. Per *i)* la (2.10) implica la (2.7) e quindi la (2.5). Poiché la (2.6) è conseguenza diretta della (2.5), la tesi è provata.

§3 *Alcuni operatori di convoluzione.* —Ricordiamo che una distribuzione $\theta \in S'(R_n)$ si dice un *moltiplicatore di Fourier* in L^p se risulta:

$$\|F^{-1}\theta F u\|_{0,p} < C\|u\|_{0,p} \qquad \forall u \in S(R^n)$$

con C indipendente da u.

Sussiste il seguente Teorma di MIHILIN (cfr.[11], [17]):

3.1 —*Condizione sufficiente perchè* $\theta(\xi)$ *sia un moltiplicatore di Fourier è che esista una costante* $k>0$ *tale che per ogni* $\alpha=(\alpha_1,\cdots,\alpha_n)$ *con* $\alpha_j \in \{0,1\}$ *si abbia:*

$$|\xi^\alpha\|D^\alpha\theta(\xi)| \leq K.$$

Introduciamo ora, per $s>0$ e $\forall j \in \{1,\cdots,n\}$, le seguenti distribuzioni temperate:

$$(3.1) \qquad \mu_{s_j}^{(j)}(x) = F^{-1}_{\xi \to x}\left[(1+\xi_j^2)^{\sigma_j/2}\xi_j^{[s_j]}\right]$$

o, equivalentemente:

$$(3.2) \qquad \mu_{s_j}^{(j)}(x) = F^{-1}_{\xi_j \to x_j}\left((1+\xi_j^2)^{\sigma_j/2}\ \xi_j^{[s_j]} F_{x_j \to \xi_j}\right) \otimes \delta(x^j),$$

dove $x^j = (\cdots x_{j-1}, x_{j+1}\cdots) \in R^{n-1}$ e $\otimes$ denota il prodotto tensoriale tra distribuzioni.

Dimostriamo che:

3.2 —*Per ogni* $u \in H^{s,p}(R^n)$ $(s>0)$ *sono equivalenti le norme:*

$$\|u\|_{s,p}, \qquad \sum_{j=1}^{n}\left\|\mu_{s_j}^{(j)} * u\right\|_{0,p} + \|u\|_{0,p}.$$

Dim. Sia $u \in S(R^n)$. Osserviamo dapprima che le funzioni:

$$\psi_0(\xi) = \sum_{j=1}^{n} \frac{(1+\xi_j^2)^{s_j/2}}{i + \sum_{j=1}^{n} (1+\xi_j^2)^{\sigma_j/2} \xi_j^{[s_j]}}$$

$$\psi_j(\xi) = \frac{(1+\xi_j^2)^{\sigma_j/2} \xi_j^{[s_j]}}{\sum_{j=1}^{n} (1+\xi_j^2)^{s_j/2}} \quad , \; j \in \{1, \cdots, n\}$$

sono, per il Teorema di Mihilin, dei moltiplicatori di Fourier; pertanto abbiamo:

(3.3)
$$\left\| F^{-1} \sum_{j=1}^{n} (1+\xi_j^2)^{s_j/2} Fu \right\|_{0,p} =$$

$$\left\| F^{-1} \psi_0(\xi) \left(i + \sum_{j=1}^{n} (1+\xi_j^2)^{\sigma_j/2} \xi_j^{[s_j]}\right) Fu \right\|_{0,p}$$

$$\leq C_1 \left(\|u\|_{0,p} + \left\| F^{-1} \sum_{j=1}^{n} (1+\xi_j^2)^{\sigma_j/2} \xi_j^{[s_j]} Fu \right\|_{0,p} \right)$$

$$\leq C_1 \left(\sum_{j=1}^{n} \left\| \mu_{s_j}^{(j)} * u \right\|_{0,p} + \|u\|_{0,p} \right).$$

Inoltre, $\forall j \in \{1, \cdots, n\}$, risulta:

(3.4)
$$\left\| \mu_{s_j}^{(j)} * u \right\|_{0,p} = \left\| F^{-1} (1+\xi_j^2)^{\sigma_j/2} \xi_j^{[s_j]} Fu \right\|_{0,p}$$

$$= \left\| F^{-1} \psi_j(\xi) \sum_{j=1}^{n} (1+\xi_j^2)^{s_j/2} Fu \right\|_{0,p}$$

$$\leq C_2 \left\| F^{-1} \sum_{j=1}^{n} (1+\xi_j^2)^{s_j/2} Fu \right\|_{0,p}$$

con C_1 e C_2 costanti positive indipendenti da u. Dalle (3.3) e (3.4) e dalla proposizione di isomorfismo 1.1 si trae l'asserto.

Se ora A e B sono due operatori, denotiamo con $[A,B]$ il commutatore $AB-BA$. Indichiamo poi con $\mu_{s_j}^{(j)}$ l'operatore:

$$(3.5) \qquad \mu_{s_j}^{(j)}: u(x)\in C_0^\infty(R^n)\to\mu_{s_j}^{(j)}\, u(x)=\mu_{s_j}^{(j)}*u(x)$$

$$=<\mu_{s_j}^{(j)}(y),u(x-y)>.$$

dove $<,>$ denota la dualità tra $C_0^\infty(R^n)$ e $D'(R^n)$.

Ciò posto, dimostriamo che:

3.3 — *Sia* $s>0$ *e* ψ *una funzione continua in* R^n, *a supporto compatto e appartenente a* M_s. *Allora esiste* $0\leq s'<s$, *uguale a* 0 *per* $s\in]0.1[$, *tale che:*

$$\sum_{j=1}^{n}\left\|\left[\psi,\mu_{s_j}^{(j)}\right]u\right\|_{0,p}\leq C\|\psi\|_{M_s}\|u\|_{s',p}, \qquad \forall u\in H^{s,p}(R^n),$$

dove C *è una costante indipendente da* u *e da* ψ.

Dim. Proviamo dapprima l'asserto per $n=1$. Sia $u\in C_0^\infty(R)$ e supponiamo, in primo luogo, $s\in]0,1[$. Poichè in tal caso $C_0^\infty(R)$ è denso in $M_s\cap C_0^{(0)}(R)$, è lecito supporre $\psi\in C_0^\infty(R)$. Ciò premesso, ricordando la (3.5) e tenendo conto dell'ipotesi su s, abbiamo:

$$[\psi,\mu_s]u(x)=<\mu_s(y),(\psi(x)-\psi(x-y))u(x-y)>$$

$$=<\underset{y\to\xi}{F}\mu_s(y),\ \underset{y\to\xi}{F^{-1}}\Big[(\psi(x)-\psi(x-y))u(x-y)\Big]>$$

$$=<(1+\xi^2)^{\frac{s}{2}},\ \underset{y\to\xi}{F^{-1}}\Big[(\psi(x)-\psi(x-y))u(x-y)\Big]>.$$

Essendo:

$$\underset{y\to\xi}{F^{-1}}\left[(\psi(x)-\psi(x-y))u(x-y)\right]=D_\xi\,\underset{y\to\xi}{F^{-1}}\left[\frac{(\psi(x)-\psi(x-y))}{y}u(x-y)\right],$$

otteniamo:

$$[\psi,\mu_s]u(x)=-s<\xi(1+\xi^2)^{\frac{s}{2}-1},\ \underset{y\to\xi}{F^{-1}}\left[\frac{(\psi(x)-\psi(x-y))}{y}u(x-y)\right]>$$

$$=-s<\underset{\xi\to y}{F^{-1}}\xi(1+\xi^2)^{\frac{s+1}{2}-\frac{3}{2}},\ \frac{(\psi(x)-\psi(x-y))}{y}u(x-y)>.$$

D' altra parte, poichè $s\in]0,1[$, si ha (cfr. [6] pag. 69):

$$\underset{\xi\to y}{F^{-1}}\ \xi(1+\xi^2)^{\frac{s+1}{2}-\frac{3}{2}} = C_s\ (sgn\ y)|y|^{\frac{(1-s)}{2}}K_{(s+1)/2}(|y|)$$

dove K_α è la funzione di Bessel modificata di parametro α e C_s è una costante assoluta. Inoltre risulta (cfr. [14] pag. 270, (c)):

$$K_{(s+1)/2}(|y|) = |y|^{-(s+1)/2}\ e^{-|y|}h(|y|)$$

con $h(y)$ funzione a crescenza lenta. Ne segue che:

$$[\psi,\mu_s]u(x)=-sC_s\int_R e^{-|y|}h(|y|)\ \frac{(\psi(x)-\psi(x-y))}{|y|^{s+1}}u(x-y)dy$$

da cui, per la diseguaglianza generalizzata di Minkowski (cfr.[15]), segue :

$$\left\|[\psi,\mu_s]u\right\|_{0,p} \leq C\|\psi\|_{M_s}\|u\|_{0,p} \qquad s<1 \tag{3.6}$$

e quindi l'asserto per $s<1$.

Sia, ora, $s\geq 1$ e $\psi\in M_s\cap C_0^{(0)}(R)$; con qualche calcolo si prova che:

$$\mu_s * u(x) = (-1)^{[s]} \mu_{s-[s]} * D_x^{[s]} u(x);$$

abbiamo allora:

$$(-1)^{[s]} [\psi, \mu_s] u(x) = \left(\psi(x) \mu_{s-[s]} * D_x^{[s]} u - \mu_{s-[s]} * D_x^{[s]} (\psi u) \right)(x)$$

$$= [\psi, \mu_{s-[s]}] D_x^{[s]} u - \mu_{s-[s]} * \sum_{k=1}^{[s]} \binom{[s]}{k} D_x^k \psi D_x^{[s]-k} u(x);$$

da 3.2 traiamo allora:

$$\left\| [\psi, \mu_s] u(x) \right\|_{0,p}$$

$$\leq C \left(\left\| [\psi, \mu_{s-[s]}] D_x^{[s]} u \right\|_{0,p} + \sum_{k=1}^{[s]} \left\| D_x^k \psi D_x^{[s]-k} u(x) \right\|_{s-[s],p} . \right)$$

Denotato al solito con s^* il massimo intero minore di s da 1.4 e 1.7 si ha:

$$\sum_{k=1}^{[s]} \left\| D_x^k \psi D_x^{[s]-k} u(x) \right\|_{s-[s],p} \leq C \|\psi\|_{M_s} \|u\|_{s^*,p},$$

e per la (3.6):

$$\left\| [\psi, \mu_{s-[s]}] \, D_x^{[s]} u \right\|_{0,p} \leq C \|\psi\|_{M_s} \|u\|_{s^*,p}$$

da cui l'asserto ponendo $s' = s^*$, per $n = 1$.

Supponiamo ora $n \geq 2$. Applicando il risultato precedente alla funzione di una variabile $u(\cdot, x^j)$ si ha:

$$\left\| [\psi, \mu_{s_j}^{(j)}] u(\cdot, x^j) \right\|^p \leq C \|\psi\|_{M_{s_j}}^p \|u(\cdot, x^j)\|_{s_j^*,p}^p \quad \forall j \in \{1, \cdots, n\}$$

con C costante indipendente da u. Integrando rispetto a x^j abbiamo:

$$\left\|\left[\psi,\mu_{s_j}^{(j)}\right]u\right\|_{o,p} \leq C \left\|F^{-1}(1+\xi_j^2)^{s_j^*/2}Fu\right\|_{o,p} \qquad \forall j\in\{1,\cdots,n\};$$

posto allora $s'=\max_j q_j s_j^*$, da 1.1 si trae l'asserto.

Siano $\Omega'\subset\subset\Omega$ aperti limitati di R_+^n e sia $\varphi\in C_0^\infty(\Omega)$ uguale a 1 in Ω'. Consideriamo l'operatore:

$$\omega_{s_j}^{(j)}: \quad u\in C_0^\infty(\overline{\Omega})\cap D'(R_+^n) \ \rightarrow \ \varphi(x)(\mu_{s_j}^{(j)}*\varphi(x)u)\in C_0^\infty(R_+^n).$$

Poiché, in virtù delle proposizioni 1.7 e 3.2 risulta:

$$\left\|\varphi(x)(\mu_{s_j}^{(j)}*\varphi u)(x)\right\|_{t,p}\leq C\left\|\mu_{s_j}^{(j)}*\varphi u\right\|_{t,p}$$

$$\leq C'\|\varphi u\|_{s+t,p}\leq C''\|u,\Omega\|_{s+t,p}$$

abbiamo che:

3.4 — *Per ogni $s>0$ e $t\geq 0$ l'operatore $\omega_{s_j}^{(j)}$ si prolunga in un operatore lineare e continuo:*

$$H^{s+t,p}(\Omega)\cap D'(R_+^n) \ \rightarrow \ \overset{0}{H}{}^{t,p}(R_+^n)$$

dove $\overset{0}{H}{}^{t,p}(R_+^n)$ denota il completamento di $C_0^\infty(R_+^n)$ rispetto alla norma di $H^{t,p}(R^n)$.

Dalle proposizioni 3.2 e 3.3 deduciamo che:

3.5 — *Per ogni $s>0$ e per ogni $u\in H^{s,p}(\Omega)\cap D'(R_+^n)$ esiste un numero non negativo $s'<s$, uguale a 0 se $s<1$, tale che:*

$$\|u,\Omega'\|_{s,p}\leq C\left(\|u,\Omega\|_{s',p}+\sum_{j=1}^{n}\left\|\omega_{s_j}^{(j)}u\right\|_{0,p}\right)$$

C essendo indipendente da u.

Dim. Dalla proposizione 3.2 si deduce che:

$$\|u,\Omega'\|_{s,p} \leq \|\chi u\|_{s,p} \leq C\left\{\|\chi u\|_{0,p} + \sum_{j=1}^{n}\left\|\mu_{s_j}^{(j)} * (\chi u)\right\|_{0,p}\right\}.$$

dove $\chi \in C_0^\infty(\Omega)$ è uguale a 1 in Ω'. Ciò posto, se supponiamo che χ sia tale che la funzione $\varphi(x)$ sia uguale a 1 sul supporto di χ, abbiamo:

$$\mu_{s_j}^{(j)} * (\chi u) = \mu_{s_j}^{(j)} * (\chi\varphi u) = \chi(\mu_{s_j}^{(j)} * (\varphi u)) + \left[\mu_{s_j}^{(j)}, \chi\right](\varphi u) =$$

$$\chi(\omega_{s_j}^{(j)} u\ ) * \left[\mu_{s_j}^{(j)}, \chi\right](\varphi u).$$

Per le proposizioni 1.7 e 3.3 abbiamo che esiste un $s' \geq 0$ e minore di s tale che:

$$\left\|\mu_{s_j}^{(j)} * (\chi u)\right\|_{0,p} \leq \left\|\omega_{s_j}^{(j)} u\right\|_{0,p} + \|\varphi u\|_{s',p}.$$

Ciò prova l'asserto.

Utilizzando la proposizione 3.3 si prova che:

3.6 — *Sia $s>0$ e ψ una fuzione continua appartenente a M_s. Allora esiste $0 \leq s' < s$, uguale a zero se $s \in]0,1[$, tale che:*

$$\left\|\left[\psi, \omega_{s_j}^{(j)}\right]u\right\|_{0,p} \leq C\|u,\Omega\|_{s',p} \quad \forall \psi \in M_s,\ \forall u \in H^{s,p}(\Omega),\ \forall j \in \{1,\cdots,n\},$$

con C costante indipendente da u.

Più generalmente se α è un multiindice tale che $<\alpha,q> < s$, esiste un $s' < s$, tale che:

$$\left\|\left[\psi D^\alpha, \omega_{s_j}^{(j)}\right]u\right\|_{0,p} \leq C\|u,\Omega\|_{s'+<\alpha,q>,p} \qquad \forall u \in H^{s,p}(\Omega).$$

§4 — *Perturbazioni singolari negli spazi* $H^{s,p}(\Omega)$. Sia Ω un aperto limitato di R^n e sia $\psi \in C_0^\infty(R_+^n)$ una funzione uguale a 1 su Ω. Dimostriamo che:

4.1 — *Nelle stesse ipotesi della proposizione* 2.1 *con* $s=0$ *e se inoltre per i coefficienti dell'operatore* $A(x,D)$ *risulta:*

$$\psi a_\alpha(x) \in M_{s+m-<\alpha,q>},\ s>0$$

allora si ha:

$$f_\epsilon \in H^{s,p}(\Omega) \Rightarrow u_\epsilon \in H^{s+m,p}(\Omega'),\quad \forall \Omega' \subset\subset \Omega.$$

Dim. Supponiamo inizialmente $s<d=\min_{<\alpha,q><m} m-<\alpha,q>$. Sia $x_0 \in \Omega'$; consideriamo l'operatore:

$$A'(x,D)=\psi(x)(A(x,D)-A_0(x_0,D))+A_0(x_0,D).$$

Se l'oscillazione dei coefficienti della parte principale A_0 dell'operatore A su Ω è sufficientemente piccola (come non è restrittivo supporre), $A'(x,D)$ verifica le ipotesi (1), (2), (3) del §2. Detta poi $\chi(x) \in C_0^\infty(\Omega)$ una funzione uguale a 1 su Ω', abbiamo che:

$$\mathcal{A}'_\epsilon(x,D)(\chi u_\epsilon)= (\epsilon^m A'(x,D)+I)(\chi u_\epsilon)=$$
$$\chi f_\epsilon+\epsilon^m[A(x,D),\chi](u_\epsilon)=h_\epsilon.$$

Se $f_\epsilon \in H^{s,p}(\Omega)$, abbiamo che $\chi f_\epsilon \in H^{s,p}(R_+^n)$; poichè risulta $\chi u_\epsilon \in H^{m,p}(R_+^n)$, dalla proposizione 1.4 segue anche che:

$$[A(x,D),\chi](u_\epsilon) \in H^{d,p}(R_+^n) \subseteq H^{s,p}(R_+^n).$$

Abbiamo dunque $h_\epsilon \in H^{s,p}(\Omega)$ e quindi, per la proposizione 2.1, $\chi u_\epsilon \in H^{s+m,p}(\Omega)$. L'asserto è cosi provato per $s<d$. La

dimostrazione si completa ragionando per induzione su s.
Possiamo ora dimostrare il:

TEOREMA 4.2 — *Nelle stesse ipotesi della proposizione* 4.1 *e se inoltre risulta:*

$$\lim_{\epsilon \to 0} \| f - f_{\epsilon}, \Omega \|_{s,p} = 0 \tag{4.1}$$

allora, per ogni $\Omega' \subset\subset \Omega$, *si ha* :

$$\lim_{\epsilon \to 0} \| f - u_{\epsilon}, \Omega' \|_{s,p} = 0 \tag{4.2}$$

$$\lim_{\epsilon \to 0} \epsilon^{m} \| u_{\epsilon}, \Omega' \|_{s+m,p} = 0. \tag{4.3}$$

Dim. Supponiamo preliminarmente $s<\min\{1,d\}$. Sia φ la funzione introdotta nel §3. Dalla proposizione 4.1 segue che $u_{\epsilon} \in H^{m+s,p}(\Omega'')$ con Ω'' contenente il supporto di φ; dalla proposizione 3.4 segue allora che $\omega_{s_j}^{(j)} u_{\epsilon} \in \overset{0}{H}{}^{m,p}(R^n_+)$.
Possiamo pertanto considerare $\mathcal{A}_{\epsilon}(x,D)(\omega_{s_j}^{(j)} u_{\epsilon})$. Abbiamo:

$$\mathcal{A}_{\epsilon}(x,D)(\omega_{s_j}^{(j)} u_{\epsilon}) = \omega_{s_j}^{(j)} f_{\epsilon} + \epsilon^{m} \left[A, \omega_{s_j}^{(j)} \right] u_{\epsilon} = l_{\epsilon}. \tag{4.4}$$

Per le ipotesi poste e per la proposizione 3.4 risulta $\omega_{s_j}^{(j)} f_{\epsilon} \to \omega_{s_j}^{(j)} f$ in $L^{p}(R^n_+)$, per $\epsilon \to 0$; inoltre per la proposizione 3.6, nel caso $s \in]0,1[$ risulta:

$$\left\| \left[A_0, \omega_{s_j}^{(j)} \right] u_{\epsilon} \right\|_{0,p} \leq C \| u_{\epsilon}, R^n_+ \|_{m,p}; \tag{4.5}$$

dalle proposizioni 1.4 e 3.4 traiamo invece:

$$\left\| \left[A - A_0, \omega_{s_j}^{(j)} \right] u_{\epsilon} \right\|_{0,p} \leq C \| u_{\epsilon}, R^n_+ \|_{m-d+s,p}; \tag{4.6}$$

da (4.4), (4.5) e (4.6), nonchè dalla proposizione 3.4 si deduce che:

$$\lim_{\varepsilon \to 0} l_\varepsilon = \omega_{s_j}^{(j)} f \quad \text{in } L^p(R^n_+);$$

Da ciò segue che la funzione $\omega_{\varepsilon_j}^{(j)} u_\varepsilon$ è soluzione del problema di perturbazioni singolari (2.1) con l_ε al posto di f_ε e dati al bordo nulli. Applicando allora il Teorema 2.2 otteniamo:

$$\lim_{\varepsilon \to 0} \omega_{s_j}^{(j)} (f - u_\varepsilon) = 0 \quad \text{in } L^p(R^n_+),$$

e quindi, per la proposizione 3.5 con $s<1$, nonché il Teorema 2.2, si giunge alle (4.2) e (4.3). La dimostrazione si completa poi ragionando per induzione su s.

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(**) Università Monte S. Angelo, Dipartimento di Matematica e applicazione
Via Cintia, 80126 Napoli - Italy

UNA BREVE NOTA SUL MONTANTE STOCASTICO

Nota di Maria Rosaria Simonelli
Presentata dal Socio Guido Trombetti
Adunanza del 2 / 3 / 91

Riassunto. E' noto in letteratura il montante a due tassi di interesse. Qui se ne vuole dare una versione stocastica.

Parole Chiavi: Semimartingale, montante a due intensità d'interesse.

§**1.** *Presentazione.* In una descrizione deterministica un progetto finanziario può essere definito da tre funzioni reali integrabili k, r e G con l'intesa che:

k(t) rappresenti il valore all'istante t dell'intensità d'interesse sui capitali presi a prestito;

r(t) rappresenti il valore all'istante t dell'intensità d'interesse sui capitali investiti;

G(t) sia la "cumulata di cassa" all'istante t ottenuta sulla base dei soli versamenti e prelievi (*flussi di cassa*) effettuati nell'intervallo temporale [0,t] senza tener conto degli interessi maturati nel medesimo intervallo.

Si supporrà naturalmente che le funzioni k,r siano positive.

La valutazione del progetto si effettua tramite il *montante*. Nel presente lavoro si definisce e si estende al caso stocastico l'equazione integrale

(6) $$S(t)=G(t)+\int_0^t S^+(s)\,k(s)\,ds-\int_0^t S^-(s)\,r(s)\,ds$$

la quale generalizza, al caso continuo e per cumulate di cassa anche non assolutamente continue, l'equazione che definisce il montante *a due tassi* descritto, nel caso discreto, nel lavoro di Teichroew D.-Robichek A.A.- Montalbano M. [6]. Per questa ragione S(t) è chiamato montante di tipo TRM. Quest'equazione esprime il montante al tempo t di un'operazione finanziaria caratterizzata dalla cumulata di cassa G(t) e dalle intensità d'interesse di finanziamento k(t) e di investimento r(t). Le funzioni k(t), r(t) sono soluzioni di due equazioni differenziali che descrivono le condizioni del mercato finanziario.

La (6) si estende al caso aleatorio sostituendo le due intensità di interesse r(t), k(t) con due *processi stocastici di interessi di finanziamento* x e *d'investimento* y del tipo di Ornstein-Uhlenbeck.

§**2.** *Descrizione del mercato finanziario.* I valori k(t), r(t) delle intensità di interesse k ed r (di mercato) possono essere "previsti" risolvendo le due equazioni differenziali:

(1) $$dk(u)=\alpha_1(\gamma_1-k(u))\,du,\quad \alpha_1>0,\quad \gamma_1>0,$$
$$dr(u)=\alpha_2(\gamma_2-r(u))\,du,\quad \alpha_2>0,\ \gamma_2>0.$$

Le costanti γ_1, γ_2 (positive) rappresentano le *intensità di interesse normale di lungo periodo* di finanziamento ed investimento e le costanti α_1, α_2 *le rapidità di aggiustamento* delle intensità d'interesse k(t), r(t) alle intensità d'interesse normali di lungo periodo γ_1, γ_2 vanno valutate a partire da osservazioni su progetti finanziari divisi in classi omogenee per importi e durata (cfr.[2] p. 236). Siano k(0), r(0) i valori osservati delle intensità d'interesse k(·) ed r(·) all'istante iniziale; allora dalle (1) segue

$$k(t) = \gamma_1 + (k(0) - \gamma_1)\, e^{-\alpha_1 t},$$
$$r(t) = \gamma_2 + (r(0) - \gamma_2)\, e^{-\alpha_2 t}.$$

Per tener conto della aleatorietà del mercato finanziario sostituiamo le funzioni deterministiche k,r con due processi stocastici x,y su uno spazio probabilizzato $(\Omega,\mathcal{Q},P)$ sul quale sia dato un processo di Wiener W, che pensiamo come soluzioni di due equazioni differenziali stocastiche ottenute introducendo un termine aleatorio di disturbo nelle equazioni (1). Pertanto adottiamo il modello di Vasicek (cfr.[7] e [2] p.267) per i *processi stocastici delle intensità d'interesse di finanziamento* x e *di investimento* y:

(2)
$$dx(t) = \alpha_1 (\gamma_1 - x(t))\, dt + \sigma_1\, dW, \qquad x(0)=k(0),$$
$$dy(t) = \alpha_2 (\gamma_2 - y(t))\, dt + \sigma_2\, dW, \qquad y(0)=r(0),$$

dove $\sigma_i>0$, $i=1,2$ e le costanti

$$\alpha_1, \alpha_2, \gamma_1, \gamma_2, k(0), r(0)$$

sono uguali alle costanti usate nel caso deterministico nelle equazioni differenziali (1). Le costanti γ_1, γ_2, sono le intensità di interesse *normale di finanziamento e di investimento di lungo periodo*; le α_i sono le *rapidità di aggiustamento* delle intensità medie $E[x(t)]=k(t)$, $E[y(t)]=r(t)$ alle intensità d'interesse normale di lungo periodo γ_1, γ_2).

Se si pone nelle (2)

$$X(t)=(x(t)- \gamma_1), \qquad Y(t)=(y(t)- \gamma_2)$$

esse diventano

(3)
$$dX(t)=-\alpha_1 X\, dt + \sigma_1\, dW, \qquad X(0)=(k(0)- \gamma_1),$$
$$dY(t)=-\alpha_2 Y\, dt + \sigma_2\, dW, \qquad Y(0)=(r(0)- \gamma_2),$$

Ognuna delle semimartingale (3) è un processo di Ornstein-Uhlenbeck (cfr. [1] p. 134 ed anche, per il teorema generale di esistenza e unicità della soluzione, [4] teor. 7, p. 197) ed è possibile fornire un'espressione esplicita dei processi soluzione

$$X(t)= e^{-t\alpha_1} [X(0)\ dt+\sigma_1 \int_0^t e^{s\alpha_1}\ dW(s)],$$
$$Y(t)= e^{-t\alpha_2} [Y(0)\ dt+\sigma_2 \int_0^t e^{s\alpha_2}\ dW(s)].$$

Descriviamo i conti per il processo X. Applicando la formula di integrazione per parti per semimartingale al processo deterministico $(t,\omega)\rightarrow e^{t\alpha_1}$ ed alla semimartingala X (cfr. [5] p. 56 la (12.1)), si ha (la componente di Ito del processo deterministico $e^{t\alpha_1}$ è 0)

$$X(t)\ e^{t\alpha_1} = X(0) + \int_0^t X(s)\ d\, e^{s\alpha_1} + \int_0^t e^{s\alpha_1}\ dX(s) =$$
$$= X(0) + \int_0^t X(s)\ \alpha_1\ e^{s\alpha_1}\ ds + \int_0^t e^{s\alpha_1}\ dX(s).$$

L'integrale stocastico di $e^{s\alpha_1}$ rispetto alla semimartingala X è dato da (cfr.[5] p.53 la (11.2))

$$\int_0^t e^{s\alpha_1}\ dX(s) = - \int_0^t e^{s\alpha_1}\ \alpha_1\ X(s)\ ds + \int_0^t e^{s\alpha_1}\ \sigma_1\ dW(s),$$

sostituendo nella formula di integrazione per parti si ha

$$X(t)\ e^{t\alpha_1}=X(0)+\int_0^t X(s)\ \alpha_1\, e^{s\alpha_1}\ ds-\alpha_1 \int_0^t e^{s\alpha_1}\ X(s)\ ds+\sigma_1 \int_0^t e^{s\alpha_1}\ dW(s)$$

semplificando

$$X(t)\ e^{t\alpha_1} = X(0) + \sigma_1 \int_0^t e^{s\alpha_1}\ dW(s)$$

e ricavando X(t)

$$X(t) = e^{-t\alpha_1} [X(0) + \sigma_1 \int_0^t e^{s\alpha_1}\ dW(s)]. \qquad (4)$$

Questo processo è detto processo di Ornstein-Uhlenbeck di parametri α_1,σ_1 uscente dal punto iniziale X(0) e relativo all'assegnato processo di Wiener W. Esso è un processo gaussiano con traiettorie continue. In particolare per ogni t la variabile aleatoria X(t) ha legge normale. I parametri di questa legge, ossia la media e la varianza di X(t), si possono facilmente calcolare tenendo conto del fatto che, per le proprietà dell'integrale di Ito (cfr. [5]), la variabile aleatoria

$$\int_0^t e^{s\alpha_1}\, dW(s)$$

ha media nulla e varianza uguale a

$$\int_0^t (e^{s\alpha_1})^2\, ds= (e^{2t\alpha_1}-1)/(2\alpha_1).$$

Si trova così:

$$E[X(t)]=X(0)\, e^{-t\alpha_1}\,,$$

$$Var[X(t)] = E[(X(t)-E[X(t)])^2] =$$
$$= E[(\sigma_1 \int_0^t e^{-(t-s)\alpha_1}\, dW(s))^2] = \sigma_1^2\ (1-e^{-2t\alpha_1})/(2\alpha_1).$$

Sostituendo $X(t)=(x(t)-\gamma_1)$ nella (4) si ha

$$x(t)=\gamma_1+e^{-t\alpha_1}\,(k(0)-\gamma_1) +\sigma_1 \int_0^t e^{-(t-s)\alpha_1}\, dW(s),$$

e con passaggi analoghi si ha

$$y(t) = \gamma_2 + e^{-t\alpha_2}\,(r(0)- \gamma_2) + \sigma_2 \int_0^t e^{-(t-s)\alpha_2}\, dW(s),$$

Si osservi che i valori medi di x(t),y(t) sono descritti dalle equazioni differenziali deterministiche (1)

$$E[x(t)] = \gamma_1 + e^{-t\alpha_1}\,(k(0)- \gamma_1),$$
$$E[y(t)] = \gamma_2 + e^{-t\alpha_2}\,(r(0)- \gamma_2).$$

In accordo col fatto che a γ_1, γ_2 abbiamo dato il nome di intensità di interesse normale di interesse di lungo periodo si ha

$$E[x(\infty)]= \lim_{t\to\infty}\ E[x(t)] = \gamma_1;$$
$$E[y(\infty)]= \lim_{t\to\infty}\ E[y(t)] = \gamma_2.$$

La varianza di x(t) ed y(t) sono uguali a quelle di X(t),Y(t):

$$Var[x(t)] = \sigma_1^2 \, (1-e^{-2t\alpha_1})/(2\alpha_1);$$
$$Var[y(t)] = \sigma_2^2 \, (1-e^{-2t\alpha_2})/(2\alpha_2).$$

Le variabili aleatorie dei processi x,y possono assumere valori negativi ma De Felice-Moriconi (cfr. [2] p. 267) assicurano che per valori "sufficientemente elevati" delle costanti $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ la probabilità di avere un'intensità negativa è piccola in ogni istante con effetti trascurabili sulla significatività economica del modello.

D'altra parte una discussione rigorosa, sui processi che rappresentano le intensità d'interesse di finanziamento x e di investimento y deve assegnare ad ognuno di questi processi come spazio degli stati $[0,+\infty[$ con uno stato riflettente nello zero, il che richiede il ricorso a due processi di Ornstein-Uhlenbeck definiti come in (3) ma con spazio degli stati $[-\gamma_1,+\infty[$, $[-\gamma_2,+\infty[$ con barriere riflettenti in $-\gamma_1,-\gamma_2$. Ad esempio, riferiamoci al primo processo definito dalle (3). Se indichiamo, gli stati del processo di Ornstein-Uhlenbeck X con s, $s\in[-\gamma_1,+\infty[$, la densità relativa allo stato s all'istante t si può ottenere risolvendo l'equazione differenziale di Fokker-Planck (cfr., per esempio, [3])

$$\partial/\partial t \, f(s,t) = \partial/\partial s \, \{ \, \alpha_1 \, s \, f(s,t) + (1/2) \, [\, \partial/\partial s \, (\, \sigma_1 \, f(s,t) \,] \, \} \, ,$$

con la condizione "iniziale"

$$\lim_{t\downarrow 0} \, f(s,t) = \delta(s-(k(0)-\gamma_1))$$

(dove con δ indichiamo la funzione che vale 1 in $k(0)-\gamma_1$ e 0 altrove)

e la condizione "al contorno"

(5) $$\lim_{s\downarrow -\gamma_1} \{\alpha_1 s \, f(s,t) + (1/2) \, (\partial/\partial s) \, [\sigma_1 \, f(s,t)] \, \} = 0.$$

La condizione iniziale esprime la circostanza che il processo X all'istante t=0 assume il valore $k(0)-\gamma_1$ con probabilità 1, cioé il processo stocastico dell'intensità d'interesse x(t), al tempo iniziale t=0 , con probabilità 1, assume il valore k(0). La condizione al contorno è un modo di esprimere la conservazione della massa di probabilità nell'intervallo $[-\gamma_1,+\infty[$ ossia la normalizzazione della densità di probabilità f il cui integrale esteso all'insieme di tutti gli stati possibili, $[-\gamma_1,+\infty[$, deve valere 1 ad ogni istante:

$$\int_{-\gamma_1}^{\infty} f(s,t)ds=1, \qquad \forall\ t\geq 0.$$

Derivando ambo i membri di questa equazione rispetto a t e ricordando l'equazione di Fokker-Planck otteniamo poi:

$$\int_{-\gamma_1}^{\infty} (\partial/\partial s \{ \alpha_1 s\ f(s,t) + (1/2) [\partial/\partial s (\sigma_1\ f(s,t)] \})\ ds =0,$$

da cui si ottiene la condizione al contorno (5).

Va sottolineato che il modello intende essere semplicemente rappresentativo delle variazioni aleatorie del mercato finanziario così come espresse mediante le semimartingale x,y allo scopo di analizzare in prima approssimazione le ripercussioni di questa aleatorietà sulla valutazione di un progetto finanziario.

§3. *Montante* (valutazione dei progetti). Una generalizzazione del *montante a due tassi,* descritta (per il caso discreto) in [6], è la funzione che indichiamo con S (boreliana) che verifichi in ciascun punto t la relazione seguente:

$$\text{(6)} \qquad S(t)=G(t)+\int_0^t S^+(s)\ k(s)\ ds -\int_0^t S^-(s)\ r(s)\ ds.$$

In questa equazione S^+, S^- denotano rispettivamente la *parte positiva* e la *parte negativa* di S definite al solito modo:

$$S^+=\sup(S,0), \qquad S^-=\sup(-S,0).$$

S(t) rappresenta il *montante* (deterministico) *a due intensità* *d'interesse* k,r *di tipo* TRM.

E' chiaro che sostituendo le funzioni deterministiche k,r con i due processi stocastici x,y l'equazione che governa la funzione montante S si trasforma nell'equazione stocastica:

$$S(t)= G(t)+\int_0^t S^+(u)\, x(u)\, du -\int_0^t S^-(u)\, y(u)\, du, \tag{7}$$

dove

$$x(u)= \gamma_1 +e^{-u\alpha_1} (k(0)- \gamma_1) +\sigma_1 \int_0^u e^{-\alpha_1 (u-s)}\, dW(s),$$
$$y(u)= \gamma_2 +e^{-u\alpha_2} (r(0)- \gamma_2) +\sigma_2 \int_0^u e^{-\alpha_2 (u-s)}\, dW(s),$$

sono due variabili aleatorie normali.

Si tratta di un'equazione stocastica molto semplice in quanto non interviene nessun differenziale del tipo di Ito ma soltanto il differenziale ds. In realtà la (7) è un'equazione differenziale nella quale i coefficienti (e quindi la soluzione) dipendono dai valori assunti dalle due variabili aleatorie normali x(u), y(u), cioè dal parametro $\omega\in\Omega$.

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ON THE OPTIMALITY CONDITIONS FOR THE EUCLIDEAN MULTIFACILITY LOCATION PROBLEM IN A TREE.

Nota di Giancarlo Pesamosca

Presentata dal Socio prof. Guido Trombetti

Adunanza del 6/4/1991

Riassunto.

Si deducono le condizioni necessarie e sufficienti di ottimalita' per il problema EMFL, per mezzo del sottodifferenziale della funzione obiettivo. Successivamente si dimostra che, nel caso particolare di un albero, tali condizioni sono equivalenti alle condizioni di ottimalita' di m problemi di Weber, con m numero di 'New Facilities' del problema EMFL.

Abstract.

The necessary and sufficient optimality conditions for the general EMFL problem are deduced by means of the subdifferential of the objective function. Moreover the equivalence between these optimality conditions and the optimality conditions for m Weber problems (with m number of new facilities) is proved for the particular case of the EMFL problem in a tree.

Key words: Location problems, Optimality conditions, Subdifferential.

1. Introduction.

The "Euclidean Multifacility Location Problem" (EMFL problem) can be outlined as follows.

A connection scheme defined by a graph H is given, connecting m unknown points $X_1, X_2, \ldots X_m$ (new facilities) and n assigned distinct points $A_{m+1}, A_{m+2}, \ldots A_{m+n}$ (existing facilities) on the euclidean plane. A positive weight is associated to every line of H. It is required to locate the new facilities so as to minimize the sum of the weighted distances among some of the existing and new facilities.

The EMFL problem has been recently considered by Conn, Calamai and Dax (e.g. see [1], [2], [4]), who deduced the optimality conditions and developed an iterative second order method for the numerical solution.

In the present paper the optimality conditions are first deduced in a quite different way (sect.3). Successively the particular case of H being a tree is considered (sect.4). In this case a simple way for testing the optimality is given, and the equivalence between the optimality conditions for the EMFL problem and for m simple Weber problems is shown.

The subject is treated in R^2, but any result can be easily extended to R^n.

2. The problem formulation.

The EMFL problem in R^2 can be formulated as follows. Find a point $X=[X_1,X_2,..X_m]^T \in R^{2m}$ which minimizes the function

$$F(X) = \sum_{\substack{(j,r)\in\mathcal{P} \\ 1\le j<r\le m}} f_{jr} + \sum_{\substack{(j,r)\in\mathcal{P} \\ 1\le j\le m,\ m+1\le r\le m+n}} h_{jr} \qquad (1)$$

where:

m is the number of new facilities,

n is the number of existing facilities,

$\mathcal{P}$ is the set of pairs $(j_1r_1),(j_2r_2),...$(considered in the lexicographic order) such that the line connecting X_j,X_r (or X_j,A_r) exists in H,

$X_j\equiv[x_j,y_j]^T$ is the location of the j-th new facility $(j=1,..m)$,

$A_j\equiv[a_j,b_j]^T$ is the location of the j-th existing facility $(j=m+1,...m+n)$,

$f_{jr}=w_{jr}\|X_j-X_r\|_2$ is the cost of the line s_{jr} connecting X_j to X_r,

$h_{jr}=\bar{w}_{jr}\|X_j-A_r\|_2$ is the cost of the line a_{jr} connecting X_j to A_r,

$w_{jr}>o$ is the weight associated with the line s_{jr},

$\bar{w}_{jr}>o$ is the weight associated with the line a_{jr}.

According to [1], in considering a current point $\bar{X}=[\bar{X}_1,\bar{X}_2,..\bar{X}_m]^T \in R^{2m}$ we will think of each new facility as being in one of the following distinct categories.

(1)-Isolated points. A new facility $\bar{X}_p$ in this category occupies a location that differs from all other facility locations.

(2)-Coinciding points. A new facility $\bar{X}_p$ belonging to this category coincides with an existing facility A_d, but differs from all other facilities.

(3)-Isolated cluster. An isolated cluster is a group of coinciding new facilities whose common location is distinct from all other facility locations.

(4)-Coinciding clusters. A coinciding cluster is a group of new facilities whose common location coincides with an with an existing facility A_d .

As a consequence the set of the m new facilities can be partitioned into t consecutive groups, each of them belonging to one of the previous categories only. The new facilities of a cluster will be considered (without loss of generality) as numbered consecutively, that is $\bar{X}_p,\bar{X}_{p+1},..\bar{X}_q$.

3. The optimality conditions for the EMFL problem.

It has been proved in [5] that the subdifferentials of the functions

$$g_1(x,y)=w[(x-a)^2+(y-b)^2]^{1/2}$$
$$g_2(x_1,y_1,x_2,y_2)=w[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$$

respectively in $X=[a,b]^T$ and in any point $X=[x_1y_1x_2y_2]^T$ such that $x_1=x_2$ and $y_1=y_2$, are given by the following sets

$$\partial g_1(a,b) = \{ x^*,y^* \mid (x^*)^2+(y^*)^2\le w^2 \} \quad (2)$$

$$\partial g_2(x_1=x_2,y_1=y_2)=\{x_1^*,y_1^*,x_2^*,y_2^* \mid x_1^*=-x_2^*,y_1^*=-y_2^*,\ (x_1^*)^2+(y_1^*)^2\le w^2\} \quad (3)$$

Therefore, if $\bar{X}_j=A_d(a_d,b_d)$ and the line a_{jd} exists in H

then it follows from (2):

$$\partial h_{jd}\Big|_{\substack{x_j=a_d\\ y_j=b_d}} = [u_{jd},v_{jd}]^T, \quad \text{with} \quad u_{jd}^2+v_{jd}^2 \le \bar{\omega}_{jd}^2 \tag{4}$$

Moreover, if $\bar{X}_j=\bar{X}_r$ and the line s_{jr} exists in H then it follows from (3):

$$\partial f_{jr}\Big|_{\substack{x_j=x_r\\ y_j=y_r}} = [u_{jr},v_{jr},-u_{jr},-v_{jr}]^T, \quad \text{with} \quad u_{jr}^2+v_{jr}^2 \le w_{jr}^2 \tag{5}$$

In the following, the first two components of ∂f_{jr} will be denoted with the positive sign if $j<r$, with the negative sign if $j>r$.

The subdifferential of the convex function $F(X)$ in a given point $\bar{X}$ will be denoted as $\partial F(\bar{X}) = [x_1^*,y_1^*,x_2^*,y_2^*,..x_m^*,y_m^*]^T$. Each pair of components $[x_j^*,y_j^*]^T$ depends on the lines of H incident to $\bar{X}_j$ only, and taking (4) and (5) into account can be written in the following way

$$\begin{bmatrix} x_j^* \\ y_j^* \end{bmatrix} = \begin{bmatrix} g_{jx} \\ g_{jy} \end{bmatrix} - \sum_{r=p}^{j-1} \begin{bmatrix} u_{jr} \\ v_{jr} \end{bmatrix} + \sum_{r=j+1}^{q} \begin{bmatrix} u_{jr} \\ v_{jr} \end{bmatrix} + \begin{bmatrix} u_{jd} \\ v_{jd} \end{bmatrix} \tag{6}$$

where:

i)- g_{jx}, g_{jy} are the partial derivatives in $\bar{X}$ (with respect to x_j and y_j) of the differentiable part of $F(X)$. If $\mathcal{R}_j$ denotes the index set of all facilities connected in H through one line to X_j, but not superimposed on $\bar{X}_j$, then such derivatives can be written as

$$g_{jx} = \sum_{\substack{r\in\mathcal{R}_j\\ r\le m}} \frac{\partial f_{jr}}{\partial x_j} + \sum_{\substack{r\in\mathcal{R}_j\\ r>m}} \frac{\partial h_{jr}}{\partial x_j} \qquad g_{jy} = \sum_{\substack{r\in\mathcal{R}_j\\ r\le m}} \frac{\partial f_{jr}}{\partial y_j} + \sum_{\substack{r\in\mathcal{R}_j\\ r>m}} \frac{\partial h_{jr}}{\partial y_j} \tag{7}$$

Observe that (7) is the only term in (6), if $\bar{X}_j$ is an isolated point.

ii)-The second and third terms appear in (6) when $\bar{X}_j$ is in a cluster. The vectors $[u_{jr},v_{jr}]^T$ are associated with the lines s_{jr}, with $\bar{X}_r$ also in the cluster. They are arbitrary, provided that the following inequalities are satisfied:

$$u_{jr}^2 + v_{jr}^2 \le w_{jr}^2 \tag{8}$$

iii)- The last term appears in the cases of coinciding points and coinciding clusters only. It is associated with the line a_{jd} connecting X_j with A_d, and must satisfy the inequality

$$u_{jd}^2 + v_{jd}^2 \leq \bar{w}_{jd}^2 \tag{9}$$

The necessary and sufficient conditions for $\bar{X}$ to be a minimizer of $F(X)$ are expressed by the equality $\partial F(\bar{X})=o$ which, tacking (6) into account, can be easily written as the linear system

$$A^*(2m\times 2c)\ W(2c\times 1) = -G(2m\times 1) \tag{10}$$

associated with the inequalities (8) and (9), for each pair (j,r) or (j,d) such that $\bar{X}_j$ is superimposed on $\bar{X}_r$ or A_d.

In the system (10):

i)-The components of the unknown vector W are the elements u_{jk}, v_{jk} of (6), considered in lexicographic order (i.e. with j increasing from 1 to m, and for each j, with k increasing from $j+1$ to the greatest index of the vertices in the cluster of $\bar{X}_j$). If c denotes the number of pairs (u_{jk}, v_{jk}), then W has $2c$ components.

ii)-A^* can be wiewed as a block matrix, whose blocks are $I(2\times 2)$ (identity matrix), $-I(2\times 2)$, and $O(2\times 2)$. In fact, each even row of A^* equals the upper odd row shifted one position to the right, and the rows 2p-1, 2p corresponding to any isolated point $\bar{X}_p$ are zero.

iii)- The vector at the right hand side is $G=[g_{1x}, g_{1y}, \ldots g_{mx}, g_{my}]^T$.

It follows from ii) that (10) can be split into two systems:

$$AU = -G_x \tag{11}$$
$$AV = -G_y \tag{11'}$$

with $U(c\times 1)$ and $V(c\times 1)$ containing respectively the u_{jk}'s and v_{jk}'s in the same lexicographic order as they are in W. The coefficient matrix $A(m\times c)$ can be obtained from A^* by substituting respectively the blocks I, $-I$, o with 1, -1, 0, and the right hand vectors are

$$G_x(m\times 1)=[g_{1x},\dots g_{mx}]^T, \qquad G_y(m\times 1)=[g_{1y},\dots g_{my}]^T.$$

Let us now consider the new facilities partitioned into t groups as shown in sect.2. In the i-th group, let m_i be the number of new facilities and c_i the number of lines s_{jr} and a_{jd}, with $\bar{X}_j, \bar{X}_r, A_d$ in the group $(m_1+..+m_t=m, c_1+..+c_t=c)$. Thus (11) can be written as

$$\begin{bmatrix} A_1 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & A_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & A_t \end{bmatrix} \begin{bmatrix} U_1 \\ U_i \\ \cdots \\ U_t \end{bmatrix} = \begin{bmatrix} -G_{1x} \\ -G_{ix} \\ \cdots \\ -G_{tx} \end{bmatrix} \tag{12}$$

and (11') likewise. So they are equivalent to the following $2t$ indipendent systems:

$$A_i(m_i\times c_i).U_i(c_i\times 1) = -G_{ix}(m_i\times 1) \qquad (i=1,\dots t) \tag{13}$$

$$A_i(m_i\times c_i).V_i(c_i\times 1) = -G_{iy}(m_i\times 1) \qquad (i=1,\dots t) \tag{13'}$$

where A_i is the submatrix constructed with the m_i rows of indices $p, p+1,..q$ of A , and with the c_i consecutive columns associated with the pairs u_{jk}, v_{jk} of the i-th group. Moreover, the elements of U_i, V_i are precisely these u_{jk} and v_{jk}, and the right hand sides are the vectors

$$G_{ix}=[g_{px}, g_{p+1,x},\dots g_{qx}]^T \qquad G_{iy}=[g_{py}, g_{p+1,y}\dots g_{qy}]^T$$

A point $\bar{X}$ solves problem (1) iff a solution U,V of systems (13), (13') exist in $\bar{X}$, satisfying the constraints (8),(9). We can now examine in detail the systems (13) and (13') for each category introduced in sect.2. In the case of clusters, the notation H_{pq} will indicate the subgraph of H containing only the facilities in the cluster, and those lines s_{jr} and a_{jd} wich interconnect them.

Isolated points.
If $\bar{X}_p$ is an isolated point, then the corresponding systems (13), (13') reduce to a single equation with left hand side zero, and giving the gradient condition in $\bar{X}_p$:

$$g_{px}=0 \; . \qquad g_{py}=0. \tag{14}$$

Coinciding points.
If $\bar{X}_p$ is superimposed on A_d, then $A_i(1\times 1)=1$, and $[u_{pd}, v_{pd}]= -[g_{px}, g_{py}]$. Thus the condition (9) can be written as

$$\varepsilon_{px}^2 + \varepsilon_{py}^2 \le \bar{w}_{pd}^2 \qquad (15)$$

Isolated clusters.
In this case, the matrix A_i of (13),(13') is the incidence matrix of the subgraph H_{pq}. Its rows correspond to the vertices $\bar{X}_j$, and its columns to the lines s_{jr} wich interconnect them. A_i has the following properties (e.g.see [3]).

1)-The sum of its rows is zero.

2)-Let A_i^o be the matrix A_i, with one row removed (reduced incidence matrix), and let M be a minor of maximal order m_i-r formed with an arbitrary set of m_i-r columns of A_i^o. Then $M\neq o$ if the columns correspond to a spanning tree of H_{pq}. Otherwise $M=o$.

If H_{pq} is a tree, then $c_i=m_i-r$, and the compatibily of systems (13), (13') requires that

$$\sum_{j=p}^{q} \varepsilon_{jx} = o \qquad \sum_{j=p}^{q} \varepsilon_{jy} = o \qquad (16)$$

If (16) are satisfied then the systems can be solved after suppressing an arbitrary row, since two systems of c_i equations in c_i unknowns, with non-zero determinant for the above property 2), are obtained in this way. As a consequence the c_i inequalities (8) can be written, which, together with the pair of equalities (16), furnish the m_i necessary and sufficient optimality conditions for the cluster.

If H_{pq} has cycles, then $c_i\geq m_i$. Since the rank of A_i is m_i-r, the compatibility of systems (13) and (13') again requires that the equalities (16) are satisfied, but in this case there are infinite solutions (see the example at the end of this section).

Coinciding Clusters.
If H_{pq} is a tree, then we can restrict ourselves to consider the case of A_d being a leaf (for example, see the cluster T in fig.1b). In fact, if A_d has degree q>1 then H_{pq} can be split down - in order to study the optimality - into q subtrees of type of Fig.1b, obtained from H_{pq} by breaking off the lines incident at A_d. In the cases of

Fig.1b, the coefficient matrix A_i is the incidence matrix of $H_{pq}-\{A_d,a_{td}\}$ with one additional column associated to the line a_{td}. Moreover $c_i=m_i$, and systems (13),(13') have m_i equations and m_i unknowns, with det $A_i\neq o$. In fact, the computation of det A_i along the column associated with a_{td} easily shows that det $A_i\neq o$, since it is equal to a minor of maximal order of the incidence matrix of a tree (see [3] again). As a consequence a single solution exists, which allows us to write the necessary and sufficient optimality conditions for the cluster. In particular the unknowns u_{td},v_{td} can be easily computed by summing the equations. So the corresponding condition (9) is obtained:

$$\left[\sum_{j=p}^{q} g_{jx}\right]^2 + \left[\sum_{j=p}^{q} g_{jy}\right]^2 \leq \bar{w}_{td}^2 \tag{17}$$

If H_{pq} has cycles then $c_i>m_i$, and the systems (13), (13') have infinite solutions as in the case of isolate clusters.

To summarize, if the connection scheme defined by the graph H is a tree then all systems (13),(13') corresponding to a given point $\bar{X}$ have a single solution, which allows us to write immediately the necessary and sufficient optimality conditions for the problem (1). If, on the contrary, H contains cycles, then some clusters might also contain cycles, so that the corresponding systems (13),(13'), (8), (9) have to be studied in order to test the optimality (see example below). Clearly, the existence of at least one solution of (13) [or (13')] satisfying the inequalities (8), (9) is required for the optimality.

Example.

The following example is considered in [4]. Let $A_4=(\sqrt{3}/2,1/2)$, $A_5=(-\sqrt{3}/2,1/2)$, $A_6=(o,-1)$ denote the locations of the existing facilities, and let

$$\begin{aligned} F(X_1,X_2,X_3) = {} & \|X_1-A_4\| + \|X_2-A_5\| + \|X_3-A_6\| \\ & + \nu(\ \|X_1-X_2\| + \|X_1-X_3\| + \|X_2-X_3\|\ ) \end{aligned} \tag{18}$$

be the objective function to be minimized (Fig.1a). The optimality of the isolated cluster $X_1=X_2=X_3=o$ is to be tested.

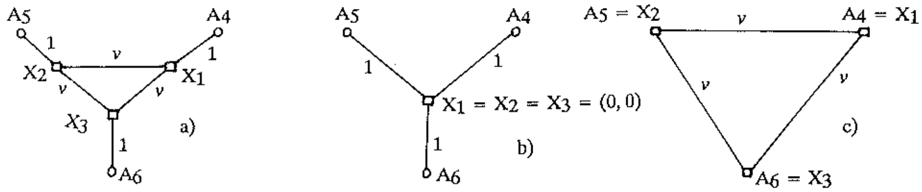

Fig.1

The system (13), (13'), (8) is written as

$$\begin{cases} u_{12}+u_{13} = -g_{1x} \\ -u_{12}+u_{23} = -g_{2x} \\ -u_{13}-u_{23} = -g_{3x} \end{cases} \qquad \begin{cases} v_{12}+v_{13} = -g_{1y} \\ -v_{12}+v_{23} = -g_{2y} \\ -v_{13}-v_{23} = -g_{3y} \end{cases} \tag{19}$$

$$\begin{cases} u^2_{12} + v^2_{12} \le \nu^2 \\ u^2_{13} + v^2_{13} \le \nu^2 \\ u^2_{23} + v^2_{23} \le \nu^2 \end{cases} \tag{20}$$

with $g_{1x}=-\sqrt{3}/2$, $g_{2x}=\sqrt{3}/2$, $g_{3x}=0$, $g_{1y}=g_{2y}=-1/2$, $g_{3y}=1$. Since the compatibility conditions (16) are satisfied, the systems (19) can be solved. The general solutions can be written as

$$U = [\ \alpha+\sqrt{3}/2,\ -\alpha,\ \alpha]^T \qquad V = [\ \beta+1/2,\ -\beta,\ 1+\beta]^T \tag{21}$$

with α, β arbitrary. By substituting (21) in (20) we obtain:

$$\begin{cases} (\alpha+\sqrt{3}/2)^2 + (\beta+1/2)^2 \le \nu^2 \\ \alpha^2 + \beta^2 \le \nu^2 \\ \alpha^2 + (1+\beta)^2 \le \nu^2 \end{cases}$$

which represents in the plane (α,β) the circles C_1,C_2,C_3 of radius ν and centres in $(-\sqrt{3}/2,-1/2)$, $(0,0)$, $(0,-1)$. As it can be easily verified, they have a common intersection for $\nu\ge\nu_0=1/\sqrt{3}$, and hence in this case the cluster $X_1=X_2=X_3=(0,0)$ solves the problem (fig.1b). On the contrary, $C_1\cap C_2\cap C_3=\emptyset$ for $\nu<\nu_0$, and the cluster can not solve the problem. The solution in this case is that shown in Fig.1c, as can be proved by simple applications of (15). For $\nu=\nu_0$ all points $X_i=tA_{i+3}$ $(i=1,2,3;\ t\in[0,1])$ solve the problem (18).

4. The particular case of trees.

If the graph H is a tree then all systems (13), (13') corresponding to a given point $\bar{X}$ have an unique solution.

In order to compute it let us consider the tree H_{pq} of the i-th cluster as directed, and denote it briefly by T. Moreover:

i)-Either set the root of T in any of its vertices, if T corresponds to an isolated cluster, or in A_d if T corresponds to a coinciding cluster. Consider T as directed from the root to the leaves.

ii)-Renumber the new facilities of T as $X_1, X_2, \dots X_{m_i}$, so that X_1 is the root for the isolated clusters, and is the successive vertex of A_d for the coinciding clusters. In this new numeration, the inequality $j<r$ must hold for every arc s_{jr} directed from X_j to X_r (see Fig.2a for an isolated cluster, and Fig. 2b for a coinciding cluster).

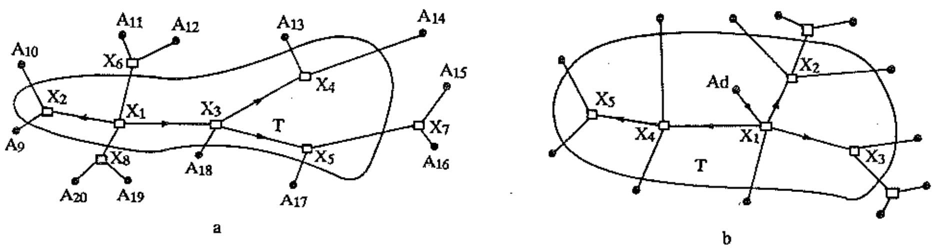

Fig.2

iii)-For any vertex X_j of T denote as X_{P_j} and $\mathcal{S}_j$ respectively the predecessor vertex of X_j, and the set of indices of the successive vertices.

Theorem 1. The solutions of systems (13), (13') can be computed by means of the following recursive formulas:

$$u_{P_j j} = g_{jx}, \quad v_{P_j j} = g_{jy} \qquad \text{if } X_j \text{ is a leaf of } T \tag{22}$$

$$u_{P_j j} = \sum_{k\in\mathcal{S}_j} u_{jk}+g_{jx}, \quad v_{P_j j} = \sum_{k\in\mathcal{S}_j} v_{jk}+g_{jy} \qquad \text{Otherwise} \tag{23}$$

(with $j=m_i,\dots 3,2$ for the isolated clusters, and $j=m_i,\dots 2,1$ for the coinciding clusters)

Proof. Consider first the case of isolated clusters, and suppose that the compatibility conditions (16) are satisfied and the equations corresponding to the root of T are removed from the systems. If X_j is a leaf, the corresponding equation (of system (13) for example) is $-u_{P_j j}=-g_{jx}$, from which (22) follows.

Otherwise, the corresponding equation is

$$-u_{P_j j} + \sum_{k\in\mathscr{S}_j} u_{jk} = -g_{jx}$$

and hence (23) also holds. The proof does not change in the case of coinciding clusters. Observe that (16) is not required in this case, and that the last step of the recursion gives u_{1d}, v_{1d} which (taking the renumeration into account) equal the sums at the left hand side of (17).□

For example, the solution of system (13) in the case of fig. 2a is given by $u_{12}=g_{2x}$, $u_{34}=g_{4x}$, $u_{35}=g_{5x}$ $u_{13}=$ $=g_{4x}+g_{5x}+g_{3x}$.

If $m=1$, the EMFL problem reduces to the Weber problem [5]:

$$F(X) = F(x,y) = \sum_{j=1}^{n} f_j = \min \qquad f_j = w_j\|X-A_j\|_2 \tag{24}$$

It follows from (14),(15) that the necessary and sufficient conditions for a given point $\bar{X}=(\bar{x},\bar{y})$ to solve problem (24) are:

$$\sum_{j=1}^{n} \frac{\partial f_j}{\partial x} = 0 \qquad \sum_{j=1}^{n} \frac{\partial f_j}{\partial y} = 0 \qquad \text{if } \bar{X}\neq A_j,\ \forall\ j \tag{25}$$

$$\left[\ \sum_{\substack{j=1\\ j\neq d}}^{n} \frac{\partial f_j}{\partial x}\right]^2 + \left[\ \sum_{\substack{j=1\\ j\neq d}}^{n} \frac{\partial f_j}{\partial y}\right]^2 \leq w_d^2 \qquad \text{if } \bar{X}=A_d \tag{25'}$$

with the derivatives computed in $\bar{X}$.

The necessary and sufficient optimality conditions for the EMFL problem in a tree H can be expressed as optimality conditions of m Weber problems (m_i problems for each i-th group). To this end, let us introduce the following notations.

T_j Subtree of T, with root in X_j and directed from X_j to the leaves of T. If X_j is a leaf, then $T_j = X_j$.

X_T Common location in $\mathbb{R}^2$ of all new facilities of T.

X_{T_j} Common location in $\mathbb{R}^2$ of all new facilities of T_j.

$\mathscr{E}$ Set of the indices of all new facilities of T.

$\mathscr{E}_j$ Set of the indices of all new facilities of T_j.

$\Gamma(X_j)$ Set of all facilities connected by one line to X_j. The new facilities lying in the group of X_j are excluded.

Moreover, let us denote a Weber problem by $W(X,\mathscr{A})$, where X is the unknown vertex, and $\mathscr{A}$ is the set of the given terminal vertices. The weights will be evident from the context.

Theorem 2. A point $\bar{X}=[\bar{X}_1,\dots\bar{X}_m]^T$ solves the EMFL problem in a tree iff:

i)-Every isolated point $\bar{X}_p$ solves the problem

$$W\left[X_p,\ \Gamma(\bar{X}_p)\right] \tag{26}$$

ii)-Every coinciding point $\bar{X}_p$ also solves problem (26).

iii)-Every i-th isolated cluster $\bar{X}_p=\bar{X}_{p+1}=\dots=\bar{X}_q$ solves the problem

$$W\left[X_T,\ \bigcup_{k\in\mathscr{E}}\Gamma(\bar{X}_k)\right] \tag{27}$$

and the m_i-1 problems

$$W\left[X_{T_r},\ \bigcup_{k\in\mathscr{E}_r}\Gamma(\bar{X}_k)\cup\{\bar{X}_{p_r}\}\right] \tag{28}$$

for $r=2,3,..m_i$ (take the renumeration of the vertices into account).

iv)-Every coiciding cluster also solves the problem (27) and the m_i-1 problems (28).

Proof. Taking (25),(25') into account, it can be easily verified that:

i)-If $\bar{X}_p$ is an isolated point, then the optimality condition for the problem (26) is given by (14).

ii)-If $\bar{X}_p$ is a coinciding point then the optimality condition for the problem (26) is given by (15).

iii)-If $\bar{X}_p=\dots=\bar{X}_q$ form an isolated cluster, then the optimality conditions for the problem (27) are given by (16), and for the m_i-1 problems (28) by the inequalities

$$u^2_{P_r r} + v^2_{P_r r} \le w^2_{P_r r} \qquad r=2,3,\ldots m_i \tag{29}$$

with

$$u_{P_r r} = \sum_{k\in\mathcal{E}_r} g_{kx} \qquad v_{P_r r} = \sum_{k\in\mathcal{E}_r} g_{ky} \tag{30}$$

But (29), (30), together with (16), are the optimality conditions for the cluster,since (29) corresponds to (8), and (30) to (22),(23) which are the solutions of systems (13), (13').

iv)-If $\bar{X}_p=\ldots=\bar{X}_q=A_d$ form a coinciding cluster, then the optimality condition for the problem (27) is clearly given by the inequality

$$\left(\sum_{j\in\mathcal{E}} g_{jx} \right)^2 + \left(\sum_{j\in\mathcal{E}} g_{jy} \right)^2 \le \bar{w}^2_{1d}$$

i.e. by (17) (taking again the renumeration into account). Moreover, the optimality conditions for the m_i-1 problems (28) are given again by (29), (30) which correspond to (8) and to (22),(23).

As a consequence, if all Weber problems (26),(27),(28) are at their minimum then pairs u_{jk}, v_{jk} exist satisfying (8),(9), and such that $\partial F(\bar{X})=0$. Therefore $\bar{X}$ is a minimizer for the EMFL problem. On the other hand, if $\bar{X}$ minimizes the EMFL problem then all components of $\partial F(\bar{X})$ vanish with all pairs u_{jk}, v_{jk} satisfying (13), (13'), (8), (9), and hence all problems (26), (27),(28) are at their minimum. □

As an example, the optimality of the cluster of fig.2a corresponds to the optimality of the following 5 Weber problems:

$W(X_1=X_2=X_3=X_4=X_5, \{A_9,A_{10},X_6,A_{13},A_{14},X_7,A_{17},A_{18},X_8\})$
$W(X_2, \{A_9,A_{10},X_1\})$
$W(X_3=X_4=X_5, \{A_{13},A_{14},X_7,A_{17},A_{18},X_1\})$
$W(X_4,\{A_{13},A_{14},X_3\})$ $\qquad W(X_5, \{A_{17},X_7,X_3\})$

Observe that these Weber problems depend on the choiche of X_1in T, so they are not univocally determined. This undetermination correspond to the undetermination of systems (13), (13') for the isolated clusters, provided that (16) is satisfied.

Conclusions and further Developments.

The main result of this paper is the expression of the optimality conditions for the EMFL problem in a tree as optimality conditions of certain m Weber problems related to the graph H. This property suggests a method for the numerical solution of the problem, based on the iterative solution of such Weber problems until they reache the optimality. This algorithm is described in [7].

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Dip. Metodi e Modelli Matematici Univ. 'La Sapienza'
Via A. Scarpa 10 00152 ROMA.

APPROXIMATION OF THE ε-SUBDIFFERENTIAL. (*)

Nota di Enrico Meccariello

Presentata dal Socio Carlo Sbordone

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Riassunto: Scopo di questo lavoro è stabilire nuovi risultati per l' approssimazione dell' ε-sottodifferenziale di una funzione convessa e semicontinua inferiormente tramite gli ε-sottodifferenziali di una successione di funzioni convesse e semicontinue inferiormente in ipotesi di Γ convergenze sequenziali o di convergenza di Mosco. In particolare si ottiene la "convergenza inferiore" degli ε-sottodifferenziali. I risultati ottenuti sono illustrati da vari esempi e si dà un' applicazione alla convergenza dei moltiplicatori di Lagrange.

Abstract: In this paper, under assumptions on Γ or Mosco convergences for a sequence of convex lower semicontinuous functions, new results about approximation for the ε-subdifferential of the limit function by the ε-subdifferential of the sequence are given.
In particular "lower convergence" of the sequence of ε-subdifferentials is obtained. In order to illustrate those results various examples and an application to the convergence of Lagrangian multipliers are given.

Key words:
Γ^- convergence of functions, sequentially lower and sequentially upper convergence of a multifunction, subdifferential, ε-subdifferential, Lagrangian multipliers.

(*) C.N.R. Fellow

§0. Introduction

Let X be a real reflexive Banach space, $X^\star$ its topological dual space and $(f_n)_n$ a sequence of *proper, convex and lower semicontinuous (l.s.c)* functions from X to $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$.

A. Ambrosetti - C. Sbordone ([1]) and M. Matzeu ([14]) were the first to give properties of convergence for a sequence of subdifferentials under assumptions of Γ^- convergence. Particularly, assuming the sequential Γ^- convergence for the sequence $(f_n)_n$ with respect to the weak convergence in X, in [14] the Graph convergence of the sequence $(\partial f_n)_n$ is obtained. A stronger and interesting property on the sequence $(\partial f_n)_n$ would be the following:

for any x in $dom f_o$ and any sequence $(x_n)_n$ weakly converging to x in X, we have: $\partial f_o(x) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial f_n(x_n)$

(that is: for any $x^\star$ in $\partial f_o(x)$ there exists a sequence $(\tilde{x}_n^\star)_n$ in $X^\star$ such that $(\tilde{x}_n^\star)_n$ strongly converges to $x^\star$ and $\tilde{x}_n^\star \in \partial f_n(x_n)$ for n large).

But as it will be shown by counter-examples, such a property is not in general obtained even for a class more restrictive than the class of convex functions.

Nevertheless by using the ε-subdifferential the following results, among the others, are obtained for $\varepsilon > 0$.

1) Under assumptions on sequential weak Γ^- convergence for the sequence $(f_n)_n$ (that is under the same assumptions than M. Matzeu) we have:

   for any $x \in dom f_o$ there exists a "good" sequence $(\tilde{x}_n)_n$ weakly converging to x in X such that: $\partial_\varepsilon f_o(x) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$ for any $\underline{\varepsilon > O}$.

2) Under a stronger assumption, that is if the sequence $(f_n)_n$ is sequentially weakly continuous converging to f_o, we obtain:

   for any $x \in dom f_o$, <u>for any sequence</u> $(x_n)_n$ weakly converging to x in

X, we have: $\partial_\varepsilon f_o(x) \subseteq s - \underset{n\to\infty}{\text{Liminf}} \partial_\varepsilon f_n(x_n)$ for any $\underline{\varepsilon > O}$

(that is the sequence of ε-subdifferentials is <u>sequentially lower convergent</u> to $\partial_\varepsilon f_o$ ([12])).

These results are compared with those of [3], [14] and various counter-examples show that it is not possible to improve them.

Then, for completeness, more obvious results of sequentially upper convergence ([12]) of ε-subdifferentials are given, extending those already obtained for subdifferentials by M. Matzeu.

Finally, an application to the convergence of Lagrangian multipliers is given in order to complete the results obtained by T. Zolezzi ([18]).

§1. Basic notions and preliminaries

First of all, let us recall some definitions of set convergence. Let (U, τ) and (V, σ) be two sequential convergence spaces ([11]) and $(A_n)_n$ a sequence of subsets of U.

Definition 1.1 ([11])

$$\tau - \underset{n\to\infty}{\text{Liminf}} A_n = \{u \in U \text{ such that there exists a sequence } (u_n)_n \; \tau\text{-converging to } u \text{ in } U, \text{ with } u_n \in A_n \text{ for } n \text{ large } \}.$$

Definition 1.2 ([11])

$$\tau - \underset{n\to\infty}{\text{Limsup}} A_n = \{u \in U \text{ such that there exists a sequence } (u_k)_k \; \tau\text{-converging to } u \text{ in } U, \text{ with } u_k \in A_{n_k} \text{ for any } k \in \mathbf{N} \text{ and for a selection of integers } (n_k)_k\}.$$

If A is a subset of X, $\overline{A}^{seq(\tau)}$ is the set of points $u \in U$ τ-limit of points of A and A is sequentially closed if and only if $A = \overline{A}^{seq(\tau)}$.

Let M be a multifunction from U to V, that is for any $u \in U$ $M(u)$ is a subset of V. Graph$M$ is the following subset of $U \times V$:

$$\text{Graph}M = \{(u,v) \in U \times V : v \in M(u)\}.$$

Now, let $(M_n)_n$ be a sequence of multifunctions from U to V for any $n \in \mathrm{N}$.

Definition 1.3 ([12])

The sequence $(M_n)_n$ is *sequentially lower convergent* to M_o if:

for any $u \in U$ and any sequence $(u_n)_n$ τ-converging to u in U we have:

$M_o(u) \subseteq \sigma - \underset{n\to\infty}{\text{Liminf}} M_n(u_n)$

(that is: for any $v \in M_o(u)$ there exists a sequence $(v_n)_n$ σ-converging to v in V such that $v_n \in M_n(u_n)$ for n large).

Definition 1.4 ([12])

The sequence $(M_n)_n$ is *sequentially upper convergent* to M_o if:

for any $u \in U$ and any sequence $(u_n)_n$ τ-converging to u in U we have:

$\sigma - \underset{n\to\infty}{\text{Limsup}} M_n(u_n) \subseteq M_o(u)$

(that is: for any sequence $(v_k)_k$ σ-converging to v in V such that $v_k \in M_{n_k}(u_{n_k})$ for any $k \in \mathrm{N}$ and for a selection of integers $(n_k)_k$ we have: $v \in M_o(u)$).

In the following we deal with a real reflexive space X and its topological dual space $X^\star$ endowed by weak and strong convergence respectively denoted by w and s.

Now let us recall some definitions of convergence for functions. Let $(f_n)_n$ be a sequence of functions from X to $\overline{\mathrm{R}}$.

Definition 1.5 ([5], [4], [2])

$$f_o(x_o) = \Gamma^-_{seq}(w)\text{lim} f_n(x_o)$$

or the sequence $(f_n)_n$ *sequentially weakly* Γ^- *converges* to f_o at x_o if:

(1.1) for any sequence $(x_n)_n$ weakly converging to x_o in X,

$$\liminf_{n\to\infty} f_n(x_n) \geq f_o(x_o)$$

(1.2) there exists a sequence $(\tilde{x}_n)_n$ weakly converging to x_o in X such that:

$$\limsup_{n\to\infty} f_n(\tilde{x}_n) \leq f_o(x_o).$$

If $f_o(x_o) = \Gamma^-_{seq}(w)\lim f_n(x_o)$ for any $x_o \in X$ we denote $f_o = \Gamma^-_{seq}(w)\lim f_n$.

Definition 1.6 ([15])

$$f_o(x_o) = \Gamma^-_{seq}(w,s)\lim f_n(x_o)$$

or the sequence $(f_n)_n$ *Mosco converges* to f_o at x_o if:

(1.1) for any sequence $(x_n)_n$ weakly converging to x_o in X,

$$\liminf_{n\to\infty} f_n(x_n) \geq f_o(x_o)$$

(1.3) there exists a sequence $(\tilde{x}_n)_n$ strongly converging to x_o in X such that:

$$\limsup_{n\to\infty} f_n(\tilde{x}_n) \leq f_o(x_o).$$

Definition 1.7 ([5], [16])

$$f_o(x_o) = \Gamma_{seq}(w)\lim f_n(x_o)$$

or the sequence $(f_n)_n$ *sequentially weakly continuous converges* to f_o at x_o if:

(1.4) for any sequence $(x_n)_n$ weakly converging to x_o in X,

$$f_o(x_o) = \lim_{n\to\infty} f_n(x_n).$$

For properties of Γ^- convergence see, for example, ([5], [4], [2], [15], [10], [13], ...). In the following $\Gamma_o(X)$ will be the set of *proper, convex and lower semicontinuous (l.s.c)* functions, all the considered functions will be in $\Gamma_o(X)$, f^* will be the *Young-Fenchel* trasform of f and *domf* the subset of X where f is finite ([8], [17]).

Finally we recall the theorem of M. Matzeu on the continuous dependence of the subdifferential under assumptions of sequentially weakly Γ^- convergence.

Proposition 1.1 ([14])

Let $(f_n)_n$ *be a sequence of* $\Gamma_o(X)$ *such that* $f_o = \Gamma^-_{seq}(w)\text{lim} f_n$.

Then:

the sequence $(\partial f_n)_n$ *of subdifferentials* $G(w,s)$ *converges to* ∂f_o,

that is ([1], [2]):

(1.5) for any $(x, x^\star) \in X \times X^\star$ *with* $x^\star \in \partial f_o(x)$ *there exists a sequence*

$(\tilde{x}_n, \tilde{x}^\star_n)_n \subset X \times X^\star$ *with* $\tilde{x}^\star_n \in \partial f_n(\tilde{x}_n)$ *for* n *large such that:*

the sequence $(\tilde{x}_n, \tilde{x}^\star_n)_n$ *is weakly-strongly converging to* $(x, x^\star)$

(1.6) for any selection of integers $(n_k)_k \nearrow \infty$, *any sequence*

$(x_k, x^\star_k)_k \subset X \times X^\star$ *with* $x^\star_k \in \partial f_{n_k}(x_k)$ *for any* $k \in \mathbf{N}$,

if the subsequence $(x_k, x^\star_k)_k$ *is weakly-strongly converging to*

$(x, x^\star)$ *then:* $x^\star \in \partial f_o(x)$. ♯

Definition 1.8 ([8], [9])

Let $f \in \Gamma_o(X)$, $x \in dom f$ and $\varepsilon \geq 0$.

$\partial_\varepsilon f(x)$ (ε-subdifferential of f at x) is the set of all $x^\star \in X^\star$ such that:

$f(y) \geq f(x) + \langle x^\star, y - x\rangle - \varepsilon$ for any $y \in X$.

Remark 1.1

$\partial_\varepsilon f(x)$ is a convex, closed set of $X^\star$ which reduces to $\partial f(x)$, the subdifferential of f at x, for $\varepsilon = 0$ and for any $\varepsilon \geq 0$ $\partial f(x) \subseteq \partial_\varepsilon f(x)$.

If $f \in \Gamma_o(X)$ and $\varepsilon > 0$ we have: $\partial_\varepsilon f(x)$ is never empty as it is easy to prove using Hahn Banach' s theorem.

§2. "Lower" convergence of the sequence $(\partial_\varepsilon f_n)_n$

First of all let us give a "local" result about convergence of the sequence $(\partial_\varepsilon f_n)_n$.

Proposition 2.1

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that:

(2.1) for any $x \in X$, for any sequence $(x_n)_n$ weakly converging to x in X it results: $\liminf\limits_{n\to\infty} f_n(x_n) \geq f_o(x)$.

If:

(2.2) for $x_o \in dom f_o$ there exists a sequence $(\tilde{x}_n)_n$ weakly converging to x_o in X such that: $\limsup\limits_{n\to\infty} f_n(\tilde{x}_n) \leq f_o(x_o)$

then:

(2.3) $\partial_\varepsilon f_o(x_o) \subseteq s - \operatorname*{Liminf}\limits_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$ *for any* $\underline{\varepsilon > O}$

(that is for any $x^\star \in \partial_\varepsilon f_o(x_o)$ there exists a sequence $(\tilde{x}_n^\star)_n$ strongly converging to $x^\star$ in $X^\star$ such that $\tilde{x}_n^\star \in \partial_\varepsilon f_n(\tilde{x}_n)$ for n large).

Proof:

1) First we want to prove:

(2.4) for any $x^\star \in X^\star$ there exists a sequence $(\tilde{x}_n^\star)_n$ strongly converging to $x^\star$ such that: $\limsup\limits_{n\to\infty} f_n{}^\star(\tilde{x}_n^\star) \leq f_o{}^\star(x^\star)$.

It is not possible to use the result given by H. Attouch ([2], Theorem 3.7, p. 271) but by using a part of his proof it is possible to obtain (2.4) under our assumptions. In effect:

let us consider the case in which the sequence $(f_n)_n$ is "equicoercive".

In this case for any $n \in \mathrm{N}$ there exists a point $\overline{x}_n$ such that minimize $\{f_n(x) - \langle x, x^\star\rangle\}$ on X. Therefore:

$$\liminf_{n\to\infty} \inf_{x\in X} \{f_n(x) - \langle x, x^\star\rangle\} = \liminf_{n\to\infty} \{f_n(\overline{x}_n) - \langle \overline{x}_n, x^\star\rangle\} =$$

$$\lim_{k\to\infty} \{f_{n_k}(\overline{x}_{n_k}) - \langle \overline{x}_{n_k}, x^\star\rangle\}.$$

Let $(\overline{x}_{n_{k_p}})_p$ be a subsequence of $(\overline{x}_{n_k})_k$ weakly converging to $\overline{x}$ in X. We have:

$$\liminf_{n\to\infty} \inf_{x\in X} \{f_n(x) - \langle x, x^\star\rangle\} \geq \{f_o(\overline{x}) - \langle \overline{x}, x^\star\rangle\} \geq$$

$$\geq \inf_{x\in X} \{f_o(x) - \langle x, x^\star\rangle\}.$$

Then (2.4) is obtanined with $\tilde{x}_n^\star = x^\star$ for any n.

Second case the sequence $(f_n)_n$ is not "equicoercive".

For any $\lambda > 0$ we introduce:

$$f_n^\lambda(x) = f_n(x) + \frac{\lambda}{2}||x||^2 \text{ for any } n \in \mathrm{N}.$$

Obviously, for any sequence $(x_n)_n$ weakly converging to x in X, we have:

$$\liminf_{n\to\infty} f_n^\lambda(x_n) = \liminf_{n\to\infty} \{f_n(x_n) + \frac{\lambda}{2}||x_n||^2\} \geq f_o(x) + \frac{\lambda}{2}||x||^2$$

and $f_n^\lambda \in \Gamma_o(X)$ for any n and for any $\lambda > 0$.

Now we prove that $(f_n^\lambda)_n$ is a sequence of functions "equicoercive" of $\Gamma_o(X)$ for any $\lambda > 0$. Infact under our assumptions:

there exists $r \geq 0$ such that for any $n \in \mathrm{N}$ and for any $x \in X$

it results: $f_n(x) + r(||x|| + 1) \geq 0$.

Denying the assertion we obtain the existence of a subsequence $(n(k))_k$ and $(x_k)_k$ such that: $f_{n(k)}(x_k) + k(||x_k|| + 1) < 0$.

Without restriction, we can assume the map $k \to n(k)$ to be increasing to $+\infty$. We have to consider two possible situations:

a) the sequence $(x_k)_k$ is bounded.

Let z be a weak limit point of a subsequence of this sequence. It follows:

$$f_o(z) \leq \liminf_{p\to\infty} \{-p||x_{k_p}|| - p\} = -\infty$$

which is in contradiction with $f_o \in \Gamma_o(X)$.

b) the sequence $(x_k)_k$ is not bounded. Let $(x_{k_p})_p$ be a subsequence such that $||x_{k_p}|| \to +\infty$ as $p \to +\infty$. Let: $z_p = t_p x_{k_p} + (1-t_p)\tilde{x}_p$ (where the sequence $(\tilde{x}_n)_n$ is the sequence weakly converging to $x_o \in dom f_o$ such that $\limsup_{n\to\infty} f_n(\tilde{x}_n) \le f_o(x_o)$). Chosen $t_p = \dfrac{1}{\sqrt{p}||x_{k_p} - \tilde{x}_p||}$ such that the sequence $(z_p)_p$ weakly converges at the point x_o as $p \to +\infty$. We observe that $t_k \to 0$ as $k \to +\infty$.

By convexity of the sequence $(f_n)_n$ it follows:

$$f_{n(k_p)}(z_p) \le t_p f_{n(k_p)}(x_{k_p}) + (1-t_p) f_{n(k_p)}(\tilde{x}_p) \le -\sqrt{p}\frac{||x_{k_p}||}{||x_{k_p} - \tilde{x}_p||} + \text{C}.$$

Thus:

$$f_o(x_o) \le \liminf_{p\to\infty} f_{n(k_p)}(z_p) \le -\infty \text{ again in contradiction with } f_o \in \Gamma_o(X).$$

And so $(f_n^\lambda)_n$ is a sequence of functions "equicoercive" for any $\lambda > 0$.

Infact let be $\lambda > 0$, for any $x \in X$ and for any $n \in \text{N}$ we have:

$$f_n^\lambda(x) = f_n(x) + \frac{\lambda}{2}||x||^2 \ge -r(||x|| + 1) + \frac{\lambda}{2}||x||^2.$$

Being the sequence $(f_n^\lambda)_n$, for any $\lambda > 0$, a sequence of functions "equicoercive" of $\Gamma_o(X)$ we have:

$$\text{for any } \lambda > 0 \text{ and for any } x^\star \in X^\star \ \limsup_{n\to\infty} (f_n^\lambda)^\star(x^\star) \le (f_o^\lambda)^\star(x^\star).$$

Therefore

$$\limsup_{\lambda\to 0+} \limsup_{n\to\infty} (f_n^\lambda)^\star(x^\star) \le \sup_{\lambda>0} (f_o^\lambda)^\star(x^\star)$$

But $\sup_{\lambda>0} (f_o^\lambda)^\star(x^\star) = f_o^\star(x^\star)$. So, we have:

$$\limsup_{\lambda\to 0+} \limsup_{n\to\infty} f_n^{\lambda^\star}(x^\star) \le f_o^\star(x^\star).$$

Using Corollary 1.16 of [2] we obtain:

there exists a map $n \to \lambda(n)$ decreasing to zero as n increasing to $+\infty$ such that: $\limsup_{n\to\infty} (f_n^{\lambda(n)})^\star(x^\star) \le \limsup_{\lambda\to 0+} \limsup_{n\to\infty} (f_n^\lambda)^\star(x^\star)$.

But

$$(f_n^{\lambda(n)})^\star(x^\star) = \inf_{u^\star \in X^\star} \{f_n^\star(u^\star) + \frac{1}{2\lambda(n)}||x^\star - u^\star||^2\}$$

and the function to minimize at the braces is convex, lower semicontinuous, coercive and defined on the reflexive Banach space $X^\star$.

Therefore, for any $n \in \mathrm{N}$, there exists $\tilde{x}_n^\star \in X^\star$ such that:

$$f_n^\star(\tilde{x}_n^\star) + \frac{1}{2\lambda(n)}||x^\star - \tilde{x}_n^\star||^2 = \inf_{u^\star \in X^\star}\{f_n^\star(u^\star) + \frac{1}{2\lambda(n)}||x^\star - u^\star||^2\}.$$

Thus:

$$f_o^\star(x^\star) \geq \limsup_{n\to\infty}\{f_n^\star(\tilde{x}_n^\star) + \frac{1}{2\lambda(n)}||x^\star - \tilde{x}_n^\star||^2\} \geq \limsup_{n\to\infty} f_n^\star(\tilde{x}_n^\star).$$

Moreover the sequence $(\tilde{x}_n^\star)_n$ is strongly convergent to $x^\star$, infact:

$$f_n^\star(\tilde{x}_n^\star) \geq \langle \tilde{x}_n^\star, \tilde{x}_n\rangle - f_n(\tilde{x}_n) \geq -\mathrm{C}(1 + ||\tilde{x}_n^\star||).$$

So, for n sufficiently large,

$$f_o^\star(x^\star) + 1 \geq -\mathrm{C}(1 + ||\tilde{x}_n^\star||) + \frac{1}{2\lambda(n)}||x^\star - \tilde{x}_n^\star||^2.$$

If $f_o^\star(x^\star) = +\infty$ there is nothing to prove; in effect in (2.4) we could have taken $\tilde{x}_n^\star = x^\star$ for any n. Otherwise, the above inequality clearly implies that $(\tilde{x}_n^\star)_n$ is strongly convergent to $x^\star$ (since $\lambda(n) \to 0$ as $n \to +\infty$).

So (2.4) holds.

2) Now let $\hat{\partial}_\varepsilon f_o(x_o)$ be the following set:

$$\hat{\partial}_\varepsilon f_o(x_o) = \{x^\star \in X^\star : f_o(x_o) + f_o{}^\star(x^\star) < \langle x_o, x^\star\rangle + \varepsilon\}.$$

From remark 1.1 we can deduce: $\hat{\partial}_\varepsilon f_o(x_o) \neq \emptyset$ for any $\varepsilon > 0$.

Let $x^\star \in \hat{\partial}_\varepsilon f_o(x_o)$. Then: $f_o(x_o) + f_o{}^\star(x^\star) - \langle x_o, x^\star\rangle < \varepsilon$.

By the assumption (2.2) there exists a sequence $(\tilde{x}_n)_n$ weakly converging to x_o such that: $\limsup_{n\to\infty} f_n(\tilde{x}_n) \leq f_o(x_o)$
and by (2.4) there exists a sequence $(\tilde{x}_n^\star)_n$ strongly converging to $x^\star$ such that: $\limsup_{n\to\infty} f_n{}^\star(\tilde{x}_n^\star) \leq f_o{}^\star(x^\star)$. But:

$$\limsup_{n\to\infty}[f_n(\tilde{x}_n) + f_n{}^\star(\tilde{x}_n^\star) - \langle \tilde{x}_n, \tilde{x}_n^\star\rangle] \leq f_o(x_o) + f_o{}^\star(x^\star) - \langle x_o, x^\star\rangle.$$

So, for any $\varepsilon > 0$, there exists $n(\varepsilon)$ such that:

for any $n \geq n(\varepsilon)$, $f_n(\tilde{x}_n) + f_n{}^\star(\tilde{x}_n^\star) < \langle \tilde{x}_n, \tilde{x}_n^\star\rangle + \varepsilon$.

That is for any $n \geq n(\varepsilon)$ $\tilde{x}_n^\star \in \partial_\varepsilon f_n(\tilde{x}_n)$. Therefore:

$$\hat{\partial}_\varepsilon f_o(x_o) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n) \text{ for a "good" sequence } (\tilde{x}_n)_n \text{ weakly converging to } x_o.$$

Now if $x^\star$ is such that: $f_o(x_o) + f_o{}^\star(x^\star) = \langle x_o, x^\star\rangle + \varepsilon$ we can consider the sequence $(z_k^\star)_k$ defined by $z_k^\star = \frac{1}{k}\overline{x}^\star + (1 - \frac{1}{k})x^\star$ with $\overline{x}^\star \in \hat{\partial}_\varepsilon f_o(x_o)$.

It is easy to prove that $z_k^\star \in \hat{\partial}_\varepsilon f_o(x_o)$ for any k and the sequence $(z_k^\star)_k$ is strongly convergent to $x^\star$. Then:

$$z_k^\star \in s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n) \text{ for any } k \in \mathbf{N} \text{ and therefore:}$$

$$x^\star \in \overline{s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)}^{seq(s)}.$$

But, with respect to the strong convergence,

$$\overline{s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)}^{seq(s)} = s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$$

so we have $x^\star \in s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$. ♯

Under assumption of sequential Γ^- convergence a global result can be given.

Corollary 2.1

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that $f_o = \Gamma^-_{seq}(w)\lim f_n$.

Then:

(2.5) for any $x \in dom f_o$ there exists a sequence $(\tilde{x}_n)_n$ weakly converging to x in X such that: $\partial_\varepsilon f_o(x) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$ for any $\underline{\varepsilon > O}$. ♯

<u>Remark 2.1</u>

Let us note that under the assumptions of corollary 2.1 lower convergence of $(\partial_\varepsilon f_n)_n$ to $\partial_\varepsilon f_o$ is not generally obtained. In effect:

let

$$f_o(x) = \begin{cases} -2 & \text{if } x = 0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$f_n(x) = \begin{cases} 2(nx - 2) & \text{if } x \geq 1/n \\ -nx - 1 & \text{otherwise} \end{cases}$$

It results:

> $f_n \in \Gamma_o(X)$ for any $n \geq 0$ and $\Gamma^-_{seq}\lim f_n = f_o$ but it is not true that for any $x \in dom f_o$ and for any sequence $(x_n)_n$ converging to x, we have: $\partial f_o(x) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n)$ for any $\underline{\varepsilon > O}$.

In fact, taken $(x_n)_n$ such that $x_n = \dfrac{2+\varepsilon}{2n}$ for any n, it can be proved that:

$\partial_\varepsilon f_n(x_n) = [0, 2n]$ for any n.

So the sequence $(x_n)_n$ converges to zero but:

$$s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n) = [0, +\infty[\not\supset \mathbf{R} = \partial f_o(0).$$

<u>Remark 2.2</u>

As shown by the next example it is not possible to take $\varepsilon = 0$ in (2.3) even if we take a sequence of functions <u>equistrongly convex</u> ([6]).

Let

$$f_o(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$f_n(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ nx^2 & \text{otherwise} \end{cases}$$

It is easy to prove that $\Gamma^-_{seq}\lim f_n = f_o$, $f_n \in \Gamma_o(X)$ for any $n \geq 0$, the sequence $(f_n)_n$ is <u>equistrongly convex</u> and we have:

> there exists $x \in dom f_o$ such that: for any sequence $(x_n)_n$ converging to x it results $s - \operatorname*{Liminf}_{n\to\infty} \partial f_n(x_n) \not\supset \partial f_o(x)$.

In fact for any sequence $(x_n)_n$ converging to zero we have:

$$s - \operatorname*{Liminf}_{n\to\infty} \partial f_n(x_n) = s - \operatorname*{Liminf}_{n\to\infty} \{\nabla f_n(x_n)\} \subset]-\infty, 0] = \partial f_o(0).$$

So it does not exist any sequence converging to zero such that:

$$\partial f_o(0) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial f_n(x_n).$$

Moreover sequentially lower convergence of $(\partial_\varepsilon f_n)_n$ to ∂f_o cannot be obtained. In fact it can be proved:

$$\partial_\varepsilon f_n(x) = [2n(x - \sqrt{\frac{\varepsilon}{n}}), 2n(x + \sqrt{\frac{\varepsilon}{n}})] \quad \text{if } x < -\sqrt{\frac{\varepsilon}{n-1}}.$$

So chosen a sequence $(x_n)_n$ such that $x_n = -\sqrt{\frac{\varepsilon+1}{n}}$ for any n it results:

the sequence $(x_n)_n$ converges to zero and $s - \operatorname{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n) = \emptyset$.

Hence $s - \operatorname{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n) \not\supset \partial f_o(0) =]-\infty, 0]$.

Let us note that the functions in this example belong to a rather regular kind of functions (squarewice, convex and derivable) and that the sequential Γ^-limit of a sequence of functions derivable may be not derivable even if the sequence is equistrongly convex. In fact in the above example f_o is not derivable at $x = 0$.

Remark 2.3

Using ε-subdifferentials we obtain results that it is not possible to obtain for subdifferentials. In addition, if we compare the Matzeu' s result with the corollary 2.1 then:

we have reversed the logical operators $\forall$ and $\exists$;

we have obtained a sort of "uniform" convergence for approximations of subgradients to f_o.

Now, let us consider a sequence Mosco converging to f_o for which it appears more interesting to give a global result.

Proposition 2.2

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that $f_o = \Gamma^-_{seq}(w,s)\lim f_n$.

Then:

(2.6) for any $x \in dom f_o$ there exists a sequence $(\tilde{x}_n)_n$ strongly converging to x in X such that: $\partial_\varepsilon f_o(x) \subseteq s - \operatorname{Liminf}_{n\to\infty} \partial_\varepsilon f_n(\tilde{x}_n)$ for any $\varepsilon > 0$

(2.7) *for any* $x^\star \in dom f_o{}^\star$ *there exists a sequence* $(\tilde{x}_n^\star)_n$ *strongly converging to* $x^\star$ *in* $X^\star$ *such that:* $\partial_\varepsilon f_o{}^\star(x^\star) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n{}^\star(\tilde{x}_n^\star)$ *for any* $\varepsilon > O$.

Proof:

Similar arguments to the ones of second part of the proof of proposition 2.1 can be used keeping in mind that :

$f_o = \Gamma^-_{seq}(w,s)\lim f_n$ is equivalent to $f_o{}^\star = \Gamma^-_{seq}(w,s)\lim f_n{}^\star$ ([15]). ∦

Now, let us give a condition which insures a stronger "local" property on convergence for the sequence $(\partial_\varepsilon f_n)_n$.

Proposition 2.3

Let $(f_n)_n$ *be a sequence of* $\Gamma_o(X)$ *such that:*

(2.1) *for any* $x \in X$, *for any sequence* $(x_n)_n$ *weakly converging to* x *in* X *it result:* $\liminf_{n\to\infty} f_n(x_n) \geq f_o(x)$.

If:

(2.8) $x_o \in dom f_o$ *is such that for any sequence* $(x_n)_n$ *strongly (respectively weakly) converging to* x_o *in* X *it results:* $\limsup_{n\to\infty} f_n(x_n) \leq f_o(x_o)$

then:

(2.9) $\partial_\varepsilon f_o(x_o) \subseteq s - \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n)$ *for any* $\varepsilon > O$ *and any sequence* $(x_n)_n$ *strongly (respectively weakly) converging to* x_o *in* X

(that is the sequence of ε-subdifferentials is sequentially lower convergent to $\partial_\varepsilon f_o$ at x_o ([12])).

Proof:

Let $x^\star \in \partial_\varepsilon f_o(x_o)$. As in the proof of the proposition 2.1 we have (2.4), that is:

there exists a sequence $(\tilde{x}_n^\star)_n$ strongly converging to $x^\star$ such that

$$\limsup_{n\to\infty} f_n{}^\star(\tilde{x}_n^\star) \le f_o{}^\star(x^\star).$$

But, for any sequence $(x_n)_n$ strongly (respectively weakly) converging to x_o, we have:

$$\limsup_{n\to\infty} f_n(x_n) \le f_o(x_o) \text{ and:}$$
$$\limsup_{n\to\infty}[f_n(x_n) + f_n{}^\star(\tilde{x}_n^\star) - \langle \tilde{x}_n, \tilde{x}_n^\star\rangle] \le f_o(x_o) + f_o{}^\star(x^\star) - \langle x_o, x^\star\rangle.$$

So for any sequence $(x_n)_n$ strongly (respectively weakly) converging to x we can prove that, as in proposition 2.1, there exists an integer $n(\varepsilon)$ such that for any $n \ge n(\varepsilon)$ $\tilde{x}_n^\star \in \partial_\varepsilon f_n(x_n)$ with $(\tilde{x}_n^\star)_n$ strongly converging to $x^\star$. Therefore (2.9) holds. ♯

Moreover we have the following global results:

Corollary 2.2

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that:

(2.1) for any $x \in X$, for any sequence $(x_n)_n$ weakly converging to x in X it result: $\liminf_{n\to\infty} f_n(x_n) \ge f_o(x)$.

If:

(2.10) for any $x \in dom f_o$ and any sequence $(x_n)_n$ <u>strongly converging</u> to x in X it results: $\limsup_{n\to\infty} f_n(x_n) \le f_o(x)$

then:

(2.11) $\partial_\varepsilon f_o(x) \subseteq s - \operatorname{Liminf}_{n\to\infty} \partial_\varepsilon f_n(x_n)$ *for any <u>$\varepsilon > 0$</u>, any $x \in dom f_o$ and <u>any sequence</u> $(x_n)_n$ strongly converging to x in X*

(that is the sequence of ε-subdifferentials is <u>sequentially lower convergent</u> to $\partial_\varepsilon f_o$ ([12])). ♯

Corollary 2.3

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that $f_o = \Gamma_{seq}(w)\lim f_n$.

Then:

(2.12) $\partial_\varepsilon f_o(x) \subseteq s - \underset{n\to\infty}{\text{Liminf}}\, \partial_\varepsilon f_n(x_n)$ *for any* $\underline{\varepsilon > O}$, *any* $x \in dom f_o$, *<u>any</u>*

<u>sequence</u> $(x_n)_n$ *weakly converging to* x *in* X. ∦

An interesting result, already known in finite dimensional space ([7]), is obtained if the sequence is constantly equal to f.

Corollary 2.4

If $f \in \Gamma_o(X)$ is sequentially weakly continuous then:

(2.13) $\partial_\varepsilon f(x) \subseteq s - \underset{n\to\infty}{\text{Liminf}}\, \partial_\varepsilon f(x_n)$ *for any* $\underline{\varepsilon > O}$, *any* $x \in dom f$ *and <u>any</u>*

<u>sequence</u> $(x_n)_n$ *weakly converging to* x *in* X

(that is the multifunction $\partial_\varepsilon f$ is <u>sequentially lower semicontinuous</u> ([12])).∦

<u>Remark 2.4</u>

This result is of a different type with respect to the one of A. Bronsted - R. T. Rockafellar ([3]) which approximate points of $Graph\, \partial_\varepsilon f$ by points of $Graph\, \partial f$ (dependent from ε), whereas we give a result (independent from ε and "uniform" in some sense) which is stronger than the approximation of $Graph\, \partial f$ by $Graph\, \partial_\varepsilon f$.

<u>§3. Upper convergence of the sequence $(\partial_\varepsilon f_n)_n$</u>

By adaptation of Matzeu' s results we obtain obvious global results for sequentially upper convergence of the ε-subdifferentials.

Proposition 3.1

Let $(f_n)_n$ be a sequence of $\Gamma_o(X)$ such that $f_o = \Gamma^-_{seq}(w)\lim f_n$.

Then:

(3.1) $s-\underset{n\to\infty}{\text{Limsup}}\,\partial_\varepsilon f_n(x_n)\subseteq\partial_\varepsilon f_o(x)$ *for any* $\varepsilon\geq O$, *any* $x\in dom f_o$ *and any sequence* $(x_n)_n$ *weakly converging to* x *in* X. ∥

Remark 3.1

The assert (3.1) means that the sequence $(\partial_\varepsilon f_n)_n$ is sequentially upper convergent to $\partial_\varepsilon f_o$ ([12]).

If the sequence is constantly equal to f we have:

Corollary 3.1

If $f\in\Gamma_o(X)$, *then:*

(3.2) $s-\underset{n\to\infty}{\text{Limsup}}\,\partial_\varepsilon f(x_n)\subseteq\partial_\varepsilon f(x)$ *for any* $\varepsilon\geq O$, *any* $x\in dom f_o$ *and any sequence* $(x_n)_n$ *weakly converging to* x *in* X

(that is the multifunction $\partial_\varepsilon f$ is *sequentially closed graph* ([12])). ∥

Remark 3.2

Let $(\varepsilon_n)_n$ be a sequence such that $(\varepsilon_n)_n\searrow\varepsilon$ and $(\varepsilon_n)_n\subset R^+$. Let us note that in the results of §2. and §3. $\partial_{\varepsilon_n}f_n(x_n)$ can be substituted for $\partial_\varepsilon f_n(x_n)$.

§4. Stability and ε- multipliers

We give an application of the previous results to problems of mathematical programming, in order to complete the results obtained by T. Zolezzi in [18] about approximation of the subdifferential of a marginal function.

4.1 Statement of the problem ([18])

(P_o) The original problem with perturbation $u\in\mathbf{R}^k$ consists of minimizing $f_o(x)$ for $x\in X$ with the constraints $g_{io}(x)\leq u_i$, $i=1,\ldots,k$.

(P_n) The nth approximating problem with perturbation u is defined as the original one with f_n and g_{in} instead of f_o and g_{io} respectively for $n = 1, 2, \ldots$ where: $f_n: X \to (-\infty, +\infty]$ and $g_{in}: X \to (-\infty, +\infty)$ are given sequences of functions.

The case $u = 0$ will correspond to the unperturbed problem.

We shall denote by $g_n(x)$ the vector of constraints $(g_{1n}(x), \ldots, g_{kn}(x))$, by $g_n(x) \leq u$ the inequalities $g_{in}(x) \leq u_i$ for $i = 1, \ldots, k$ and by $g_n(x) < u$ the corresponding strict inequalities.

The nth admissible region is defined by

$$D_n(u) = \{x \in X \text{ such that } g_n(x) \leq u\}.$$

As in [18] we denote the nth marginal function by

$$p_n(u) = inf\, f_n(D_n(u))$$

and by $\Omega(c)$ the corresponding set:

$$\Omega(c) = \bigcup_{n=0}^{\infty} \{x \in X \text{ such that } f_n(x) \leq c,\ g_{in}(x) \leq c,\ i = 1, \ldots, k\}$$

for any real c.

Then we have:

Proposition 4.1.1

If the following conditions are satisfied:

(4-1) $\Omega(c)$ *is sequentially weakly compact for any real* c

(4-2) $f_o = \Gamma^-_{seq}(w, s)\lim f_n$ *and* $g_o = \Gamma_{seq}(w)\lim g_n$

(4-3) for any $n \geq 0$ f_n, g_n *are convex and weakly lower semicontinuous*

(4-4) there exists $x_o \in dom f_o$ *such that* $g_o(x_o) < O$

(4–5) *there exists* $u^\star \in \mathbf{R}^k$ *with* $u^\star < 0$ *such that*

$D_o(u^\star)$ *is nonempty and* f_o *is proper on* $D_o(u^\star)$

then:

(4–6) $\operatorname*{Limsup}_{n\to\infty} \partial_\varepsilon p_n(u_n) \subseteq \partial_\varepsilon p_o(0)$ *for any* $\varepsilon \geq O$ *and any sequence* $(u_n)_n$ *converging to zero*

(4–7) $\partial_\varepsilon p_o(0) \subseteq \operatorname*{Liminf}_{n\to\infty} \partial_\varepsilon p_n(u_n)$ *for any* $\varepsilon > 0$ *for any sequence* $(u_n)_n$ *converging to zero.*

Proof:

As in [18], from (4–1) to (4–5), we deduce that:

- for any n the function p_n is proper, convex and lower semicontinuous
- $p_o = \Gamma^-_{seq} \lim p_n$.

Therefore we can apply the proposition 3.1 and the assertion (4–6) holds.

In order to prove (4–7) it is sufficient to show:

$0 \in dom\ p_o$ and for any sequence $(u_n)_n$ converging to zero it results: $\limsup_{n\to\infty} p_n(u_n) \leq p_o(0)$.

From assumption (4–4) it is easy to show that $p_o(0)$ is finite,

so $0 \in dom\ p_o$.

Then for any $\varepsilon > 0$ there exists x_ε such that:

$f_o(x_\varepsilon) < p_o(0) + \varepsilon$ and $g_o(x_\varepsilon) \leq 0$.

We have two cases to consider:

i) $g_{io}(x_\varepsilon) < 0$ for $i = 1, 2, \ldots, k$

ii) there exists $i^\star \in \{1, 2, \ldots, k\}$ such that $g_{i^\star o}(x_\varepsilon) = 0$.

In the case i), from the assumption (4–2) we have:

there exists $y_n \to x_\varepsilon$ such that $f_n(y_n) \to f_o(x_\varepsilon)$ and $g_n(y_n) \to g_o(x_\varepsilon)$.

Since the sequence $(g_n(y_n) - u_n)_n$ converges to $g_o(x_\varepsilon) < 0$ we have:

there exists $\overline{n}$ such that for $n \geq \overline{n}$ it results:

$g_n(y_n) - u_n \leq 0$ that is $y_n \in D_n(u_n)$.

But from $p_n(u_n) \leq f_n(y_n)$ we obtain $\limsup_{n\to\infty} p_n(u_n) \leq f_o(x_\varepsilon) < p_o(0) + \varepsilon$.

Now, let us suppose we are in the case ii).

Let x_o the point satisfying the assumption (4–4). Then:

– if $0 \leq f_o(x_o)$ the point:

$$z_\varepsilon = (1-\alpha)x_\varepsilon + \alpha x_o \quad \text{with} \quad 0 < \alpha < \frac{\varepsilon}{f_o(x_o) + \varepsilon + 1} \in]0,1[$$

is such that:

$f_o(z_\varepsilon) < p_o(0) + 2\varepsilon$ and $g_o(z_\varepsilon) < 0$

and by similar arguments to the ones used in the case i) we have:

$$\limsup_{n\to\infty} p_n(u_n) \leq f_o(z_\varepsilon) < p_o(0) + 2\varepsilon.$$

– If $f_o(x_o) < 0$ we can consider the following auxiliary functions:

$F_n() = f_n() - f_o(x_o)$ for any $n \in \mathbf{N}$.

Therefore there exists t_ε such that $f_o(t_\varepsilon) < p_o(0) + 2\varepsilon$ and $g_o(t_\varepsilon) < 0$.

Hence we have again $\limsup_{n\to\infty} p_n(u_n) < p_o(0) + 2\varepsilon$.

We can conclude that for any sequence $(u_n)_n$ converging to zero it results:

$\limsup_{n\to\infty} p_n(u_n) \leq p_o(0)$ and we may apply the proposition 2.3. ♯

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Università Monte S. Angelo, Dipartimento di Matematica e applicazione
Via Cintia, 80126 Napoli - Italy

AN ELEMENTARY PROOF OF A CLOSED GRAPH THEOREM ON NORMED $\mathcal{K}$-SPACE

Nota di Endre Pap and Charles Swartz
Presentata dal socio Paolo de Lucia
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Abstract. In this paper we shall give a version of the Closed Graph Theorem whose proof does not use the category arguments and which is quite elementary and need only some basic properties of the operators on normed spaces.
Riassunto. Si dà una versione del Teorema del Grafico Chiuso nella cui dimostrazione non si fa uso della teoria delle categorie di Baire.

In many introductory functional analysis texts the Closed Graph Theorem (and its close relative, the Open Mapping Theorem) are proven by using Baire Category arguments ([2], [6], [8], [9]). In this note we present a version of the Closed Graph Theorem whose proof does not rely on category arguments and uses only basic properties of normed spaces.

Throughout this note let X and Y be normed linear spaces and let $T : X \to Y$ be a linear map. Our notation and terminology is standard and will essentially follow [8]. The domain of the adjoint of T, T', is $\{y' \in Y' : y'T$ is continuous on $X\} = \mathcal{D}(T')$, and $T' : \mathcal{D}(T') \to X'$ is defined to be $T'y' = y'T$. We have the following properties of adjoints.

Proposition 1. (a) $T' : \mathcal{D}(T') \to X'$ is a closed operator.
(b) If T is closed, then $\mathcal{D}(T')$ is weak dense in Y'.

For (a), see IV. 8.3 of [8], and for (b) see IV. 8.1 of [8]. The proofs of these results are straightforward and use only standard facts from introductory functional analysis.

For our proof, we also require

Proposition 2. Let S be a linear map with domain $\mathcal{D}(S) \subseteq X$ and range in Y with Y complete. If S is bounded and closed, then $\mathfrak{D}(S)$ is (norm) closed.

Proof: Let $x \in \overline{\mathfrak{D}(S)}$ and pick $\{x_k\} \subseteq \mathfrak{D}(S)$ such that $x_k \to x$. Since $\|Sx_i - Sx_j\| \leq \|S\| \; \|x_k - x_j\|$, $\{Sx_k\}$ is cauchy and, therefore, converges to some $y \in Y$. Since S is closed, $x \in \mathfrak{D}(S)$ and $y = Sx$. Therefore, $\mathfrak{D}(S)$ is closed.

We need one further result of E. Pap on adjoint operators ([4]). A sequence $\{x_k\}$ in X is called a $\mathcal{K}$-sequence if every subsequence of $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ such that the subseries

$$\sum_{k=1}^{\infty} x_{n_k}$$

converges to an element of X. A $\mathcal{K}$-sequence obviously converges to 0, and if X is a Banach space, then any sequence which converges to 0 is a $\mathcal{K}$-sequence (If $\|x_k\| \to 0$, then given any subsequence pick a subsequence $\{x_{n_k}\}$ satisfying

$$\sum_{k=1}^{\infty} \|x_{n_k}\| < \infty \;]$$

There are, however, spaces which are not complete but have the property that a sequence $\{x_k\}$ converges to 0 if and only if it is a $\mathcal{K}$-sequence ([3]); such spaces are called $\mathcal{K}$-spaces.

Theorem 3. (Pap) If X is a -space, then $T' : \mathfrak{D}(T') \to X'$ is continuous.

For the sake of completness we shall give here a sketch of the proof. For more details see [1] and also [5] (with a different approach but also elementary).

It suffices to prove that if $\{y'_i\} \subseteq \mathfrak{D}(T')$ and $\|y'_i\| \leq 1$, then $\{T'y'_i\}$ is bounded. We take $x_i \in X$ such that $\|x_i\| = 1$ and $\|T'y'_i\| \leq \langle T'y'_i \; ,x_i \rangle + 1$.

Hence, it suffices to prove that $\{\langle T'y'_i ,x_i \rangle\}$ is bounded, i.e. that if $\{t_i\}$ is positive sequence of scalars which converges to 0, then, as $i \to \infty$) $t_i \langle T'y'_i ,x_i \rangle \to 0$. Let $z_{ij} = \langle \sqrt{t_i}\, T'y'_i , \sqrt{t_j}\, x_j \rangle$ $(i,j \in \mathbb{N})$. It is easy to see that $[z_{ij}]$ is a $\mathcal{K}$-matrix, i.e.

(I) $\lim_i z_{ij} = z_j$ exists for each j and

(II) for each subsequence $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that $\left\{ \sum_{j=1}^{\infty} z_{i n_j} \right\}$ is Cauchy.

Then the Basic Matrix Theorem from [1] , 3.11 (If $[z_{ij}]$ is a $\mathcal{K}$-matrix, then $\lim_i z_{ii} = 0$, with a quite elementary and easy proof) implies

$$t_i \langle T'y'_i, x_i \rangle \to 0 \text{ as } i \to \infty.$$

We now have the machinery in place to give our version of the Closed Graph Theorem.

Theorem 4. Let X be a $\mathcal{K}$-space and Y reflexive. If $T : X \to Y$ is closed then T is continuous.

Proof: We first show that $\mathcal{D}(T') = Y'$. By proposition 1 (b), $\mathcal{D}(T')$ is weak dense since Y is reflexive. By Theorem 3 and Proposition 1 (a), T' is closed and continuous, and, therefore, by Proposition 2, $\mathcal{D}(T')$ is norm closed. But $\mathcal{D}(T')$ is a linear subspace and, therefore, has the same closure in the norm and weak topologies ([8] III. 6.3). It follows that $\mathcal{D}(T') = Y'$.

Since $\|Tx\| = \sup\{|\langle y', Tx\rangle| : \|y'\| \leq 1\} = \sup\{|\langle T'y', x\rangle| : \|y'\| \leq 1\}$ $\leq \|x\| \sup\{\|T'y'\| : \|y'\| \leq 1\} = \|T'\| \, \|x\|$, T is continuous.

The "usual" form of the Closed Graph Theorem for normed spaces requires that the domain space X and the range space Y be complete ([2] , [6] , [8] , [9]). Thus, the version of the Closed Graph Theorem given in Theorem [4] can be viewed as a result in which the usual assumption on the domain space is relaxed from X being complete to X being a $\mathcal{K}$-space but the assumption on the range space is strengthened from completeness to reflexivity.

It should be pointed out that [7] contains a version of the Closed Graph Theorem when X is a (normed) $\mathcal{K}$-space and Y is a B-space. However, the proof of the Closed Graph Theorem in [7] is not elementary; in place of standard Baire Category Methods, it uses Kein-Smulian Theorem to show that $\mathcal{D}(T') = Y'$.

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E. Pap, Institute of Mathematics, University of Novi Sad.
C. Swartz, Department of Mathematical Sciences, New Mexico State University.

UN PROBLEMA DI CONTROLLO AI LIMITI
PER L'EQUAZIONE DELLE CORDE VIBRANTI

Nota di Angela Maria Forenza e Salvatore Giuga

Presentata dal Socio Guido Trombetti

Adunanza del 1/6/91

Abstract. In this paper a optimal control problem with quadratic cost functional for the vibrating string equation is considered.The boundary conditions are governed by the solution of a linear differential equation involving the control. A characterization of optimal control are derived.

Riassunto. In questa nota si considera un classico problema per l'equazione delle corde vibranti nel quale le condizioni laterali si esprimono tramite la soluzione di un sistema di equazioni differenziali lineari contenenti il controllo. Assegnato un funzionale quadratico(costo),dipendente dal controllo sia direttamente che tramite le soluzioni del problema lineare e del problema per l'equazione delle corde vibranti,si dimostra l'esistenza e l'unicità del minimo di tale funzionale(controllo ottimo);si stabilisce una condizione necessaria e sufficiente di ottimalità e si deduce una rappresentazione implicita del controllo ottimo.

1-Posizione del problema

Siano $A(t)=(a_{i,j}(t))$ e $B(t)=(b_{i,j}(t))$ matrici rispettivamente $m \times m$ e $m \times k$ i cui elementi sono funzioni continue nell'intervallo $[0,T]$ e sia $U=L^2(0,T;R^K)$ lo spazio delle funzioni vettoriali a k componenti a quadrato sommabili in $[0,T]$ (spazio dei controlli).Denotata con $z(t)$ una funzione vettoriale ad m componenti definita in $[0,T]$ e con $u(t)$ un elemento di U,consideriamo il sistema di equazioni differenziali lineari:

$$z' = A(t)z + B(t)u(t) \tag{1}$$

con la condizione iniziale:

$$z(0) = z_0 \tag{2}$$

Per ogni $u \in U$ il sistema (1)con la condizione (2)ammette un'unica soluzione,diciamola z_u ,rappresentata (cfr.(1)):

$$z_u(t) = \Phi(t,0)z_0 + \int_0^T \Phi(t,s)B(s)u(s)ds$$

dove Φ é la matrice di evoluzione generata dalla matrice A. Se C é una matrice costante 2 x m,per ogni $u \in U$,poniamo:

$$v(t) = \begin{pmatrix} v_1(t) \\ \\ v_2(t) \end{pmatrix} = C\int_0^t z_u(\tau)(t-\tau)d\tau \,. \tag{3}$$

Sia ora D il rettangolo delpiano (O,t,x) definito dalle limitazioni:

$$0 \leq t \leq T \;; \quad 0 \leq x \leq L \,.$$

Consideriamo (cfr.(3)) il classico problema della ricerca di una soluzione $y(t,x)$ dell'equazione

$$\Box y = 0 \quad ; \qquad \Box = \frac{\partial^2}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2}{\partial t^2} \tag{4}$$

verificante le condizioni:

$$y(t,o) = v_1(t) \quad ; \quad y(t,L) = v_2(t) \tag{5}$$

$$y(0,x) = 0 \qquad ; \quad \partial_t y(0,x) = 0 \quad (\partial_t = \partial/\partial t) \,. \tag{6}$$

Tale problema ammette in D una e una sola soluzione che si può indicare con $y_u(t,x)$ in modo da mettere in rilievo la sua dipendenza dal controllo u e che si rappresenta nella forma:

$$y_u(t,x) = \frac{2a}{L} \sum_{k=1}^{\infty} \text{sen}\, \frac{k\pi x}{L} \int_0^t (v_1(\tau)-(-1)^k v_2(\tau))\, \text{sen}\frac{a\pi k(t-\tau)}{L} d\tau$$

In conclusione, per ogni $u \in U$ la soluzione del problema(1)⟶(6) consiste nella coppia $(z_u(t),y_u(t,x))$.

2-Il problema di controllo

Associamo al problema (1)⟶(6) il seguente costo quadratico:

$$(7) \qquad \begin{aligned} J(u) = & \int_0^T\int_0^L\int_0^L K_1(t,x;\xi)\, y_u(t,\xi) y_u(t,\xi)\, dx\, d\xi\, dt + \int_0^T <E(t)u(t), \\ & u(t)>dt + \int_0^L\int_0^L K_2(x,\xi)\, \partial_t y_u(T,x)\, \partial_t y_u(T,\xi)\, dx\, d\xi + \\ & \int_0^T <H_1(t)z_u(t),z_u(t)>dt + < H_2 z_u(T),\dot{z}_u(T)> \end{aligned}$$

dove la notazione <.,.> denota il prodotto scalare e le funzioni K_1,K_2,H_1,H_2,E soddisfano le seguenti ipotesi:

a) $K_1: [0,T] \times [0,1] \times [0,1] \longrightarrow R$ é una funzione continua con le derivate parziali prime rispetto a tutte le variabili tale che $K_1(t,x;\xi) = K_1(t,\xi;x)$ ed inoltre

$$\int_0^L\int_0^L K_1(t,x,\xi)f(x)f(\xi)\,dx\,d\xi \geq 0$$

per ogni $t \in [0,T]$ e per ogni $f \in C([0,L];R)$, essendo $C([0,L];R)$ l'insieme delle funzioni reali continue in $[0,L]$.

b) $K_2: [0,L] \times [0,L] \longrightarrow R$ é una funzione continua tale che

$$\int_0^L\int_0^L K_2(x,\xi)g(x)g(\xi)\,dx\,d\xi \geq 0 \text{ e. } K_2(0,\cdot)=K_2(L,\cdot)=0.$$

c) $E(t)$ é una matrice k x k i cui elementi sono funzioni reali continue in $[0,T]$, simmetrica, cioè tale che $H(t)= H^*(t)$ con H^* matrice trasposta di H, e tale che

$$\int_0^T <E(t)h(t),h(t)>dt > 0$$

per ogni $h \in L^2(0,T;R^k)$.

d) $H_1(t)$ é una matrice m x m i cui elementi sono funzioni sommabili in $[0,T]$, simmetrica e tale che

$$\int_0^T < H_1(t)k(t),k(t) > dt \geqq 0$$

per ogni $k \in C([0,T];R^m)$;

e) H_2 é una matrice m x m ad elementi costanti, simmetrica e tale che $< H_2 k,k > \geqq 0$ per ogni costante k in R^m .

Dalle ipotesi a) $\longrightarrow$ e) consegue che, per ogni u appartenente a U, risulta $J(u) \geqq 0$ onde si pone il problema di controllo ottimo consistente nello stabilire se il funzionale J(u) é dotato di minimo in U e nel determinare gli eventuali controlli minimizzanti J(u).

3- Esistenza di un controllo ottimo

Si verifica, tenendo conto delle ipotesi poste, che il funzionale J(u) é strettamente convesso in U.

Osserviamo inoltre che l'operatore

$$Z: u \in L^2(0,T;R^k) \longrightarrow z_u(t) \in C([0,T];R^m)$$

con $C([0,T];R^m)$ munito della topologia della uniforme convergenza é continuo.

Essendo infatti

$$z_u(t) = \Phi(t,0)z_\circ + \int_0^T \Phi(t,s)B(s)u(s)ds$$

per provare l'asserto basta verificare che, posto:

$$F(u) = \int_0^t \Phi(t,s)B(s)u(s)ds$$

risulta

$$||F(u)||_{C([0,T])} \leq k\,||u||_{L^2(0,T)}$$

per ogni $u \in U$ e k costante positiva non dipendente da u.

Posto $M=\max |\phi(t,s)B(s)|$ con $t,s \in [0,T]$, si ha:

$$\left|\int_0^t \phi(t,s)B(s)u(s)ds\right| \leq M \int_0^T |u(s)|ds \leq$$

$$MT^{1/2}\left(\int |u(s)|^2 ds\right)^{1/2} \leq k \| u \|_{L^2(0,T)} \qquad (k=MT^{1/2})$$

Dunque l'operatore Z é continuo.

Essendo inoltre

$$v(t)=\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = C\int_0^t z_u(\tau)(t-\tau)d\tau$$

dalla continiutà dell'operatore Z si deduce anche la continuità dell'operatore:

$$V : u \in L^2(0,T; R^k) \longrightarrow v(t) \in C^1([0,T];R^2)$$

e cioè dell'operatore che al controllo associa i dati laterali del problema iperbolico. Poichè, come é noto, y_u e $\partial_t y_u$ dipendono con continuità dalla coppia dei dati laterali, dalla continuità dell'operatore V si deduce anche che y_u e $\partial_t y_u$ dipendono con continuità dal controllo u.

In altri termini sono continui i seguenti operatori:

$$\Lambda : u \in L^2(0,T;R^k) \longrightarrow y_u \in C([0,T] \times [0,L];R)$$

$$\Lambda' : u \in L^2(0,T;R^k) \longrightarrow \partial_t y_u \in C([0,T] \times [0,L];R)$$

Dalla continuità degli operatori Z, Λ, Λ' si deduce che il funzionale J(u) é continuo in U.

Dalle ipotesi poste si deduce infine che il funzionale J(u) é coercitivo in U.

Ne segue, in virtù del teorema di esistenza e di unicità del controllo ottimo relativo ai funzionali strettamente convessi, coercitivi e semicontinui inferiormente (cfr. (2)), l'esistenza e l'unicità del controllo ottimo $\bar{u}$, cioè del controllo $\bar{u}$ tale che $J(\bar{u})=\inf J(u)$ $\forall u \in U$.

Un semplice calcolo esprimente la necessarietà dell'annullarsi della derivata secondo Gateaux di J(u) (che nel caso di funzionali convessi é anche sufficiente)conduce alla seguente condizione necessaria e sufficiente di ottimalità:

$$
(8)\quad
\begin{aligned}
&\iint_D (y_u(t,x)-y_{\bar u}(t,x))\,dt\,dx\int_0^L K_1(t,x;\xi)y_{\bar u}(t,\xi)\,d\xi \;+\\
&\int_0^L (y_u(T,x)-y_{\bar u}(T,x))\,dx\int_0^L K_2(x;\xi)y_{\bar u}(T,\xi)\,d\xi \;+\\
&\int_0^T < z_u(t)-z_{\bar u}(t),H_1(t)z_{\bar u}(t) >dt + < z_u(T)-z_{\bar u}(T),H_2 z_{\bar u}(T)>+\\
&\int_0^T < u(t)-\bar u(t),E(t)\bar u(t) >dt = 0. \qquad (\forall\, u \in U)
\end{aligned}
$$

4- Caratterizzazione del controllo ottimo

Denotiamo con $(p(t,x),\psi(t))$ lo stato aggiunto di $(y(t,x),z(t))$ definito come soluzione del seguente problema:

$$(4^*)\qquad \Box p = \int_0^L K_1(t,x;\xi)y_{\bar u}(t,\xi)\,d\xi$$

$$(5^*)\qquad p(t,0)=0 \quad ; \; p(t,L)=0$$

$$(6^*)\qquad p(T,x)=0 \quad ; \; \partial_t p(T,x)=-a^2\int_0^L K_2(x,\xi)y_{\bar u}(T,\xi)\,d\xi$$

$$(1^*)\qquad \psi' = -A^*(t)\psi - C^*\int_T^t w(\tau)(t-\tau)\,d\tau + H_1(t)z_{\bar u}(t)$$

$$w(t) = \begin{pmatrix} \partial_x p(t,0) \\ \\ -\partial_x p(t,L) \end{pmatrix}$$

$$(2^*)\qquad \psi(T) = -H_2 z_{\bar u}(T)$$

dove $(y_{\bar u}, z_{\bar u})$ é la soluzione del problema (1)→(6) corrispondente al controllo ottimo $\bar u$.

Utilizzando questo stato aggiunto si ottiene la seguente caratterizzazione del controllo ottimo.

<u>Proposizione</u> Il controllo ottimo $\bar{u}$ si rappresenta nella forma

$$\bar{u}(t) = E^{-1}(t)B^{*}(t)\,\psi(t)$$

essendo ψ la soluzione del problema $(1^{*}),(2^{*})$.

<u>Dim</u>. Premettiamo che,nelle ipotesi poste,sussiste la seguente formula di Green's:

$$\iint_D (p(t,x)\,\Box y(t,x) - y(t,x)\,\Box p(t,x))\,dxdt =$$

$$\int_0^T (p(t,L)\,\partial_x y(t,L) - y(t,L)\,\partial_x p(t,L)) - (p(t,0)\,\partial_x y(t,0) -$$

$$y(t,0)\,\partial_x p(t,0))dt + a^{-2}\int_0^L (p(T,x)\,\partial_t y(T,x) - y(T,x)\,\partial_t p(T,x)$$

$$- (p(0,x)\,\partial_t y(0,x) - y(0,x)\,\partial_t p(0,x)))dx.$$

Applicando questa formula alla coppia $(p, y_u - y_{\bar{u}})$ con p soluzione del problema $(4^{*})\rightarrow(6^{*})$ si ottiene:

$$\iint_D (y_u(t,x) - y_{\bar{u}}(t,x))\,dxdt \int_0^L K_1(t,x;\xi)\,y_{\bar{u}}(t,\xi)\,d\xi \;+$$

$$\int_0^L (y_u(T,x) - y_{\bar{u}}(T,x))\,dx \int_0^L K_2(x,\xi)\,y_{\bar{u}}(T,x)\,d\xi \;-$$

$$\int_0^T ((v_1(t) - \bar{v}_1(t))\,\partial_x p(t,0) - (v_2(t) - \bar{v}_2(t))\,\partial_x p(t,L))\,dt = 0.$$

Conseguentemente la condizione di ottimalità (8) si può riscrivere nella forma:

$$\int_0^T \langle z_u(t) - z_{\bar{u}}(t), H_1(t) z_{\bar{u}}(t)\rangle dt + \langle z_u(T) - z_{\bar{u}}(T), H_2 z_{\bar{u}}(T)\rangle \;+$$

$$(9)\quad +\int_0^T < u(t)-\bar{u}(t), E(t)\bar{u}(t) >dt - \int_0^T < v(t)-\bar{v}(t), w(t) >dt=0.$$

D'altra parte risulta:

$$\int_0^T <v(t)-\bar{v}(t), w(t) >dt= \int_0^T <C \int_0^t (z_u(\tau)-z_{\bar{u}}(\tau))(t-\tau)d\tau; w(t)>dt$$

$$= \int_0^T d\tau \int_\tau^T <(z_u(\tau)-z_{\bar{u}}(\tau))(t-\tau), C^*w(t) >dt =$$

$$= \int_0^T <z_u(\tau)-z_{\bar{u}}(\tau) \;,\; C^* \int_T^\tau w(t)(\tau-t)dt > d\tau \quad =$$

$$= \int_0^T <z_u(\tau)-z_{\bar{u}}(\tau), \; - \psi'(\tau)-A^*(\tau)\psi(\tau)+ H_1(\tau)z_{\bar{u}}(\tau) >d\tau =$$

$$= - <z_u(T)- z_{\bar{u}}(T) \;,\; \psi(T) > + \int_0^T < (z_u-z_{\bar{u}})'(\tau), \psi(\tau)>d\tau$$

$$- \int_0^T <z_u(\tau)- z_{\bar{u}}(\tau) \;, A^*(\tau)\psi(\tau) >d\tau \; + \int_0^T <z_u(\tau)-z_{\bar{u}}(\tau) \;,$$

$$H_1(\tau)z_{\bar{u}}(\tau) >d\tau =$$

$$= - <z_u(T)- z_{\bar{u}}(T) \;,\; \psi(T) > + \int_0^T <B(\tau)(u(\tau)-\bar{u}(\tau), \psi(\tau)>d\tau$$

$$+ \int_0^T <z_u(\tau)-z_{\bar{u}}(\tau), H_1(\tau)z_{\bar{u}}(\tau) >d\tau .$$

Conseguentemente la (9) si può ancora riscrivere nella forma:

$$\int_0^T < u(t)- \bar{u}(t), E(t)\bar{u}(t) >dt - \int_0^T < u(t)- \bar{u}(t), B^*(t)\psi(t)>dt$$

$$= 0 \quad ;$$

cioè nella forma:

$$\int_0^T < u(t)- \bar{u}(t), \; E(t)\bar{u}(t)- B^*(t)\psi(t) >dt = 0 \; (\forall\, u \in U).$$

Di quì si trae:

$$E(t)\bar{u}(t) - B^*(t)\psi(t) = 0$$

e quindi l'asserto.

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SU UNA SOLUZIONE FEEDBACK IN UN PROBLEMA DI CONTROLLO PER L'EQUAZIONE DELLE CORDE VIBRANTI

Nota di Salvatore GIUGA
presentata dal Socio Guido TROMBETTI
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ABSTRACT. *In this paper a boundary control problem with quadratic cost for the vibrating string equation with boundary conditions given by the solution of a linear differential equation involving the control is considered. Using the results of [1], the existence and uniqueness of the solution both for the Hamilton type systems are discussed. The form of the feedback for the optimal control are derived.*

RIASSUNTO. In questo lavoro si considera un problema di controllo ottimo con costo quadratico per l'equazione delle corde vibranti; utilizzando i risultati di [1], si dimostra l'esistenza e l'unicità per il sistema di tipo Hamilton associato e si ricava la forma del feedback lineare per il controllo ottimo.

Introduzione In [1] A.M. FORENZA e S. GIUGA hanno dimostrato un teorema di esistenza e unicità per il problema di controllo ottimo che richiamiamo qui di seguito.

Indichiamo con: T un numero positivo; $A(t)$ e $B(t)$ matrici rispettivamente $m \times m$ e $m \times k$ con elementi in $C^{(0)}([0,T])$; z_0 un vettore di R^m. Introduciamo il sistema differenziale:

$$(I) \qquad \frac{dz}{dt} = A(t)z + B(t)u(t); \qquad z(0) = z_0$$

dove il vettore $u \in L^2(0,T;R^k)$ è il controllo del nostro problema. Siano poi: C una matrice $2 \times m$ ad elementi costanti; v_0 e v_0' due vettori di R^2. Indicata con z_u la soluzione del sistema (I), poniamo:

$$(II) \qquad v_u(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = v_0 + v_0' t + \int_0^t C z_u(\tau)(t-\tau)d\tau$$

e, detto L un numero positivo, consideriamo il seguente problema per l'equazione delle corde vibranti:

$$(III) \qquad \begin{cases} \partial_t^2 y - \partial_x^2 y = 0 \\ y(t,0) = v_1(t); \quad y(t,L) = v_2(t) \quad t \in [0,T] \\ y(0,x) = y_0(x); \quad \partial_t y(0,x) = y_0'(x); \quad x \in [0,L] \end{cases}$$

dove:$y_0(x) \in C^{(1)}([0,L])$;$y_0'(x) \in C^{(0)}([0,L])$; inoltre:

$$\begin{pmatrix} y_0(0) \\ y_0(L) \end{pmatrix} = v_0 \qquad \begin{pmatrix} y_0'(0) \\ y_0'(L) \end{pmatrix} = v_0'$$

Indicata con $y_u(t,x)$ la soluzione del problema (III), introduciamo il funzionale costo:

$$\begin{aligned} J(u) = &\int_0^T dt \int_0^L dx \int_0^L K_1(t,x;\xi) y_u(t,x) y_u(t,\xi) d\xi \\ &+ \int_0^L dx \int_0^L K_2(x,\xi) y_u(T,x) y_u(T,\xi) d\xi \\ &+ \int_0^T \langle H_1(t) z_u(t), z_u(t) \rangle \, dt + \langle H_2 z_u(T), z_u(T) \rangle \\ &+ \int_0^T \langle E(t) u(t), u(t) \rangle \, dt \end{aligned}$$

dove $<>$ denota il prodotto scalare tra vettori numerici; inoltre:

i) $K_1(t,x;\xi)$ e $K_2(x;\xi)$ sono nuclei continui, semidefiniti positivi e simmetrici rispetto a x e a ξ; inoltre risulta

$$\begin{aligned} K_1(t,0;\xi) = & K_1(t,L;\xi) = 0 \quad \forall t \in [0,T] \\ K_2(0;\xi) = & K_2(L;\xi) = 0 \end{aligned}$$

ii) $H_1(t)$ e H_2 sono matrici $m \times m$ semidefinite positive e $H_1(t)$ è a elementi continui.

iii)$E(t)$ è una matrice $k \times k$ definita positiva e ad elementi continui.

In [1] i dati v_0,v_0' , $y_0(x)$, $y_0'(x)$ sono stati supposti nulli; ciò è inessenziale ai fini del risultato, che è espresso dal:

TEOREMA I *Nelle ipotesi i), ii), iii) esiste un unico controllo* $\tilde{u} \in L^2(0,T;R^k)$ *tale che:*

$$J(\tilde{u}) \leq J(u) \qquad \forall u \in L^2(0,T;R^k)$$

In questa nota dimostro il:

TEOREMA II *Il controllo ottimo $\tilde{u}$ è rappresentabile come un feedbak lineare della forma:*

$$\begin{aligned} \tilde{u}(t) = & \int_0^L S_1(t,x) y_{\tilde{u}}(t,x) dx + \int_0^L S_2(t,x) \partial_t y_{\tilde{u}}(t,x) dx \\ & + S_3(t) v_{\tilde{u}}(t) + S_4(t) v_{\tilde{u}}'(t) + S_5(t) z_{\tilde{u}}(t) \end{aligned}$$

La dimostrazione del Teorema II è contenuta nel n.5. Nel n.1 si introduce lo stato aggiunto del sistema e viene riportata una rappresentazione del controllo ottimo ricavata in [1]; nel n.2 si introduce il sistema di tipo Hamilton e se ne dimostra l'univoca risolubilità;

nel n.3 si stabiliscono alcune formule di rappresentazione, tramite le quali, nel n. 4, il sistema di tipo Hamilton viene ricondotto ad una equazione integrale di Fredholm; ciò consente, insieme alla formula richiamata nel n. 1, di pervenire alla dimostrazione del Teorema II.

1. Lo stato aggiunto Sia $u(t)$ un controllo e $(z_u(t), y_u(t,x))$ la corrispondente soluzione del sistema (I), (II), (III).
Sia $p_u(t,x) \in C^{(1)}([0,T] \times [0,L])$ la soluzione del sistema:

$$(1.1) \qquad \begin{cases} \partial_t^2 p - \partial_x^2 p = \int_0^L K_1(t,x:\xi) y_u(t,\xi) d\xi \\ p(t,0) = p(t,L) = 0 \quad t \in [0,T] \\ p(T,x) = 0; \quad \partial_t p(T,x) = -\int_0^L K_2(x;\xi) y_u(T,\xi) d\xi \end{cases}$$

poniamo inoltre:

$$(1.2) \qquad w(t) = \begin{pmatrix} \partial_x p(t,0) \\ -\partial_x p(t,L) \end{pmatrix}$$

e, indicate con $A^*(t)$ e C^* le matrici aggiunte di $A(t)$ e C rispettivamente, consideriamo il sistema differenziale:

$$(1.3) \qquad \begin{cases} \dfrac{d\varphi}{dt} = -A^*(t)\varphi + H_1(t) z_u(t) + C^* \int_t^T (t-\tau) w(\tau) d\tau \\ \varphi(T) = -H_2 z_u(T) \end{cases}$$

la coppia (p_u, φ_u) soluzione del sistema (1.1), (1.2), (1.3) è detta stato aggiunto per il nostro problema. In [1] è stato dimostrato il:

TEOREMA III *Il vettore $u \in L^2(0,T;R^k)$ coincide con il controllo ottimo se e solo se risulta:*

$$(1.4) \qquad u(t) = E(t)^{-1} B(t)^* \varphi_u(t)$$

2. Il sistema di tipo Hamilton Siano: $s \in [0,T[$; $z_s \in R^m$; v_s,$v'_s \in R^2$; $\gamma_1(x) \in C^{(1)}([0,L])$;$\gamma_2 \in C^{(0)}([0,L])$. Nel dominio D_s=$[s,T] \times [0,L]$, consideriamo il seguente sistema, che conveniamo di indicare con $\Sigma_s(z_s, v_s, v'_s, \gamma_1, \gamma_2)$:

$$
(2.1)\qquad \begin{cases}
\dfrac{dz}{dt} = A(t)z + B(t)E^{-1}B^*(t)\varphi(t); \quad z(s) = z_s \\
\dfrac{d^2v}{dt^2} = Cz(t) \quad v(s) = v_s \quad v'(s) = v'_s \\
\partial_t^2 y - \partial_x^2 y = 0 \quad y(s,x) = \gamma_1(x) \quad \partial_t y(s,x) = \gamma_2(x) \\
y(t,0) = v_1(t) \quad y(t,L) = v_2(t) \\
\partial_t^2 p - \partial_x^2 p = \displaystyle\int_0^l K_1(t,x;\xi)y(t,\xi)d\xi \\
p(T,x) = 0 \quad \partial_t p(T,x) = -\displaystyle\int_0^L K_2(x;\xi)y(T,\xi)d\xi \\
p(t,0) = 0 \quad p(t,L) = 0 \\
\dfrac{d\varphi}{dt} = -A^*(t)\varphi + H_1(t)z(t) + \\
C^* \displaystyle\int_t^T (t-\tau) \begin{pmatrix} \partial_x p(\tau,0) \\ -\partial_x p(\tau,L) \end{pmatrix} d\tau \\
\varphi(T) = -H_2 z(T)
\end{cases}
$$

nel vettore incognito (z,v,y,p,φ). Vale la:

PROPOSIZIONE 2.1 *Il sistema:*

$$\Sigma_0(z_0, v_0, v'_0, y_0(x), y'_0(x))$$

è univocamente risolubile e la sua soluzione è il vettore

$$X_{\tilde{u}} = (z_{\tilde{u}}, v_{\tilde{u}}, y_{\tilde{u}}, p_{\tilde{u}}, \varphi_{\tilde{u}})$$

$\tilde{u}$ essendo il controllo ottimo.

DIM. Che il vettore $X_{\tilde{u}}$ sia soluzione del sistema è conseguenza diretta del Teorema III. Viceversa, se (z,v,y,p,φ) è soluzione del sistema, posto:

$$u = E^{-1}(t)B^*(t)\varphi(t) \tag{2.2}$$

si verifica subito che $z = z_u, v = v_u,\ y = y_u,\ p = p_u,\ \varphi = \varphi_u$. Dalla (2.2) e da Teorema III segue allora che $u = \tilde{u}$.

Dalla proposizione ora dimostrata discende l'altra:

PROPOSIZIONE 2.2 *Il sistema:*

$$\Sigma_s(z_{\tilde{u}}(s), v_{\tilde{u}}(s), v'_{\tilde{u}}(s), y_{\tilde{u}}(s,x), \partial_t y_{\tilde{u}}(s,x)) \tag{2.3}$$

è univocamente risolubile e la sua soluzione è la restrizione del vettore $X_{\tilde{u}}$ al dominio D_s.

3. Formule di rappresentazione Fissato $s \in [0,T]$, consideriamo, nel dominio $D = [0,T] \times [0,L]$, il problema:

$$\begin{cases} \partial_t^2 y - \partial_x^2 y = f(t,x) \\ y(t,0) = 0 \quad y(t,L) = 0 \\ y(s,x) = \gamma_1(x) \quad \partial_t y(s,x) = \gamma_2(x) \end{cases} \tag{3.1}$$

dove $\gamma_i(x) \in C^{(0)}(0,L)$, $i = 1,2$, e $f(t,x) \in C^{(0)}$.
Indichiamo con $Y_s[\gamma_1,\gamma_2,0,f]$ l'unica soluzione debole del problema (3.1). Se, assegnata una funzione $h(x)$ nell'intervallo [0,L], indichiamo con $\breve{h}(x)$ la funzione dispari, periodica di periodo 2L, coincidente con $h(x)$ in $[0,L]$, è noto che risulta:

$$Y_s[\gamma_1,\gamma_2,0,f] = \frac{1}{2}\left(\breve{\gamma}_1(x+t-s) + \breve{\gamma}_1(x-(t-s))\right) \tag{3.2}$$

$$+\frac{1}{2}\int_{x-(t-s)}^{x+t-s} \breve{\gamma}_2(\xi)\,d\xi + \frac{1}{2}\int_s^t d\tau \int_{x-(t-\tau)}^{x+t-\tau} \breve{f}(\tau,\xi)\,d\xi$$

Utilizzando sviluppi trigonometrici si perviene alla seguente altra rappresentazione:

$$Y_s[\gamma_1, \gamma_2, 0, f](t,x) = \frac{2}{L}\sum_{k=1}^{\infty} \sin k\omega x \cos k\omega(t-s) \int_0^L \gamma_1(\xi) \sin k\omega\xi \, d\xi$$

$$+\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin k\omega x \sin k\omega(t-s)}{k} \int_0^L \gamma_2(\xi)\, d\xi$$

$$+\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin k\omega x}{k} \int_s^t d\tau \int_0^L f(\tau,\xi) \sin k\omega(t-\tau) \sin k\omega(\xi)\, d\xi$$

dove si è posto $\omega = \pi/L$. Introducendo i nuclei:

$$G(t,x;\tau,\xi) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin k\omega x \sin k\omega(t-\tau) \sin k\omega\xi}{k}$$

$$G'(t,x;\tau,\xi) = \partial_t G(t,x;\tau,\xi)$$

possiamo allora scrivere:

$$Y_s[\gamma_1, \gamma_2, 0, f](t,x) = \int_0^L G'(t,x;s,\xi)\gamma_1(\xi) d\xi \tag{3.3}$$

$$+\int_0^L G(t,x;s,\xi)\gamma_2(\xi)\, d\xi + \int_s^t d\tau \int_0^L G(t,x;\tau,\xi) f(\tau,\xi)\, d\xi$$

Il nucleo G è una funzione costante a tratti; il nucleo G' è una funzione generalizzata. Entrambi poi sono simmetrici rispetto a x, ξ . Vale la seguente:

PROPOSIZIONE 3.1 *Se $h(t,x;s,\xi)$ indica $G(t,x;s,\xi)$ o $G'(t,x;s,\xi)$, allora risulta:*

$$\int_0^L h(t,\xi;s,\xi')K_1(t,x;\xi)\, d\xi \in C^{(0)}(D\times D) \tag{3.4}$$

$$\int_0^L h(t,\xi;s,\xi')K_2(x,\xi)\,d\xi \in C^{(0)}([0,L]\times D) \tag{3.5}$$

DIM. Per la simmetria del nucleo G' , da (3.2) e (3.3) traiamo:

$$\int_0^L G'(t,\xi;s,\xi')K_1(t,x;\xi)\,d\xi = \int_0^L G'(t,\xi';s,\xi)K_1(t,x;\xi)\,d\xi$$

$$= \frac{1}{2}\left(\breve{K}_1((t,x;\xi'+t-s)+\breve{K}_1(t,x;\xi'-(t-s))\right)$$

da cui la (3.4) con $h = G'$ essendo, per l'ipotesi i), $\breve{K}_1(t,x;\xi)$ continua in $D \times R$. Allo stesso modo si procede per la (3.5) e per $h = G$.

Consideriamo ora, nel dominio $D_s = [s,T] \times [0,L]$, il problema:

$$\begin{cases} \partial_t^2 y - \partial_x^2 y = 0 \\ y(s,x) = \partial_t y(s,x) = 0 \\ \begin{pmatrix} y(t,0) \\ y(t,L) \end{pmatrix} = v(t) \quad v(t) \in C^{(0)}([s,T]) \end{cases} \tag{3.6}$$

e indichiamone con $Y_s[0,0,v(t),0](t,x)$ l'unica soluzione. Introduciamo poi la matrice:

$$\theta(t,x;s) = (\theta_1(t,x;s), \theta_2(t,x;s))$$

dove:

$$\begin{aligned} \theta_1(t,x;s) &= \left(1-\frac{x}{L}\right)(t-s) - \int_0^L G(t,x;s,\xi)\left(1-\frac{\xi}{L}\right)d\xi \\ \theta_2(t,x;s) &= \frac{x}{L}(t-s) - \int_0^L G(t,x;s,\xi)\frac{\xi}{L}\,d\xi \end{aligned} \tag{3.7}$$

Si verifica immediatamente che:

$$\begin{aligned} \theta_1(t,x;s) &= Y_s[0,0,\begin{pmatrix} t-s \\ 0 \end{pmatrix},0](t,x) \\ \theta_2(t,x;s) &= Y_s[0,0,\begin{pmatrix} 0 \\ t-s \end{pmatrix},0](t,x) \end{aligned} \tag{3.8}$$

e che, posto $\theta'(t,x;s) = \partial_t \theta(t,x;s)$ risulta:

$$(3.9) \qquad \begin{aligned} \theta'_1(t,x;s) =& Y_s[0,0,\begin{pmatrix}1\\0\end{pmatrix},0](t,x) \\ \theta'_2(t,x;s) =& Y_s[0,0,\begin{pmatrix}0\\1\end{pmatrix},0](t,x) \end{aligned}$$

Una verifica diretta prova la:

PROPOSIZIONE 3.2 *Sussistono le eguaglianze:*

$$Y_s[0,0,v(s),0](t,x) = \theta'(t,x;s)v(s)$$
$$Y_s[0,0,(t-s)v'(s),0](t,x) = \theta(t,x;s)v'(s)$$
$$Y_s[0,0,\int_s^t (t-\tau)Cz(\tau)d\tau,0](t,x) = \int_s^t \theta(t,x;\tau)Cz(\tau)d\tau$$

Nel seguito applicheremo l'ultima di tali formule alla soluzione $z(t)$ del sistema differenziale (I), la quale ha l'espressione:

$$\Phi(t,s)z(s) + \int_s^t \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

dove Φ è la matrice di evoluzione di $A(t)$. Ciò posto, dalla proposizione 3.2 traiamo:

$$(3.10) \qquad Y_s[0,0,\int_s^t (t-\tau)C\phi(\tau,s)z(s)\,d\tau,0](t,x)$$
$$= \left[\int_s^t \theta(t,x;\tau)C\Phi(\tau,s)\,d\tau\right] z(s)$$

$$Y_s[0,0,\int_s^t (t-\tau)d\tau \int_s^\tau C\Phi(\tau,\sigma)B(\sigma)u(\sigma)d\sigma,0](t,x) =$$
$$\int_s^t \theta(t,x;\tau)d\tau \int_s^\tau C\Phi(\tau,\sigma)B(\sigma)u(\sigma)d\sigma =$$
$$\int_s^t \left[\int_\sigma^t \theta(t,x;\tau)C\Phi(\tau,\sigma)d\tau\right] B(\sigma)u(\sigma)d\sigma$$

Se poniamo:

$$\begin{aligned}\eta(t,x;s) &= \int_s^t \theta(t,x;\tau)C\Phi(\tau,s)d\tau \quad \tau \le t \\ \eta(t,x;s) &= 0 \quad s > t\end{aligned} \tag{3.11}$$

tali formule divengono:

$$Y_s[0,0,\int_s^t (t-\tau)C\Phi(\tau,s)z(s)\,d\tau,0](t,x) =$$

$$\eta(t,x;s)z(s)$$

$$Y_s[0,0,\int_s^t (t-\tau)\left[C\int_s^\tau \Phi(\tau,\sigma)B(\sigma)u(\sigma)d\sigma\right]d\tau,0](t,x)$$

$$= \int_s^T \eta(t,x;\tau)B(\tau)u(\tau)d\tau$$

Dimostriamo la:

PROPOSIZIONE 3.3 *Se $h(t,x;\tau)$ indica uno dei nuclei $\theta(t,x;\tau)$,$\theta'(t,x;\tau)$o $\eta(t,x;\tau)$, allora risulta:*

$$\int_0^L h(t,\xi;\tau)K_1(t,x;\xi)d\xi \in C^{(0)}(D\times[0,T]) \tag{3.12}$$

$$\int_0^L h(T,\xi;\tau)K_2(\xi,x)d\xi \in C^{(0)}([0.L]\times[0,T]) \tag{3.13}$$

DIM. Se h=θ o h=η, la tesi è immediata, essendo tali nuclei continui. Per completare la dimostrazione osserviamo che dalla prima delle (3.7) e dalla (3.2) si trae:

$$\theta_1'(t,x;\tau) = 1-\frac{x}{L}-\int_0^L G'(t,x;\tau,\xi)\left(1-\frac{\xi}{L}\right)d\xi$$

$$= 1-\frac{x}{L}-\frac{1}{2}(\psi(x+t-\tau)+\psi(x-(t-\tau))$$

dove $\psi(x)$ denota il prolungamento periodico di periodo 2L della funzione dispari coincidente con $1-\frac{x}{L}$ nell'intervallo $]0,L[$. Abbiamo dunque:

$$\int_0^L \theta_1'(t,\xi;\tau)K_1(t,x;\xi)d\xi = \int_0^L \left(1-\frac{\xi}{L}\right) K_1(t,x;\xi)d\xi$$

$$-\frac{1}{2}\int_0^L (\psi(\xi+t-\tau)+\psi(\xi-(t-\tau))K_1(t,x;\xi)d\xi$$

Entrambi gli addendi a secondo membro sono funzioni continue; in particolare il secondo addendo è uguale a:

$$-\frac{1}{2}\int_0^L \psi(\xi')(\check{K}_1(t,x;\xi'-(t-\tau))+\check{K}_1(t,x;\xi'+t-\tau))d\xi'$$

e la sua continuità discende dall'ipotesi i). In modo analogo si ragiona per θ_2'.

4. Rappresentazione dello stato aggiunto Sia $h(t,x)$ un funzionale lineare agente sulle funzioni della variabile x appartenenti a $C^{(0)}([0,L])$, dipendente dal parametro $t \in [0,T]$, tale che:

$$\int_0^L h(t,x)K_1(t,x;\xi)d\xi \in C^{(0)}(D) \tag{4.1}$$

$$\int_0^L h(T,\xi)K_2(x,\xi)d\xi \in C^{(0)}(D) \tag{4.2}$$

Sia poi $\zeta(t) \in C^{(0)}([0,T])$. Consideriamo, nel dominio D il seguente sistema, che conveniamo di indicare con $S(h,\zeta)$:

$$\begin{cases} \partial_t^2 p - \partial_x^2 p = \int_0^L h(t,\xi)K_1(t,x;\xi)d\xi \\ \quad p(t,0) = 0 \quad p(t,L) = 0 \\ \quad p(T,x) = 0 \quad \partial_t p(T,x) = -\int_0^L h(T,\xi)K_2(x,\xi)d\xi \\ \quad \frac{d\varphi}{dt} = -A^*(t)\varphi + H_1(t)\zeta(t) + \\ \qquad C^* \int_t^T (t-\tau) \begin{pmatrix} \partial_x p(\tau,0) \\ -\partial_x p(\tau,L) \end{pmatrix} d\tau \\ \quad \varphi(T) = -H_2\zeta(T) \end{cases}$$

Per le ipotesi assunte sui dati, il sistema $S(h,\zeta)$ ammette un'unica soluzione $(p(t,x),\varphi(t))$; per essa si ha:

$$p(t,x) \in C^{(1)}(D) \qquad \varphi(t) \in C^{(0)}([0,T])$$

come si deduce dalle formule di rappresentazione:

$$(4.3) \quad p(t,x) = \int_T^t d\tau \int_0^L G(t,x;\tau,\xi')d\xi' \int_0^L h(\tau,\xi)K_1(\tau,\xi';\xi)d\xi$$
$$- \int_0^L G(t,x;T,\xi')d\xi' \int_0^L h(T,\xi)K_2(\xi',\xi)d\xi$$

$$(4.4) \qquad \varphi(t) = -\Phi^*(t,T)H_2\zeta(T) + \int_T^t \Phi^*(t,\tau)H_1(\tau,\zeta(\tau)d\tau$$
$$+ \int_T^t \Phi^*(t,\tau)d\tau \int_\tau^T C^*(\tau-\sigma) \begin{pmatrix} \partial_x p(\sigma,0) \\ -\partial_x(\sigma,L) \end{pmatrix} d\sigma$$

Conviene osservare che se i dati $h(t,x)$ e $\zeta(t)$ dipendono con continuità da altri parametri, tale proprietà si trasmette alla coppia soluzione. È il caso dei nuclei introdotti nel n.3, i quali, in virtù delle proposizioni 3.1 - 3.3, verificano tutti le (4.1) e (4.2) uniformemente rispetto ai parametri da cui dipendono.

Nel seguito, la coppia soluzione del sistema $S(h,0)$ verrà denotata con:

$$\begin{aligned}
&(p_1(t,x;s,\xi),\varphi_1(t;s,\xi)) && \text{se} \quad h(t,x) = G'(t,x;s,\xi)\\
&(p_2(t,x;s,\xi),\varphi_2(t;s,\xi)) && \text{se} \quad h(t,x) = G(t,x;s,\xi)\\
&(p_3(t,x;s),\varphi_3(t;s)) && \text{se} \quad h(t,x) = \theta'(t,x;s)\\
&(p_4(t,x;s),\varphi_4(t;s)) && \text{se} \quad h(t,x) = \theta(t,x;s)
\end{aligned}$$

Infine, ponendo:

$$\begin{aligned}
\Phi_0(t,\tau) &= \Phi(t,\tau) && \text{per } \tau \le t\\
\Phi_0(t,\tau) &= 0 && \text{per } \tau > t
\end{aligned}$$

la coppia soluzione del sistema $S(\eta(t,x;s),\Phi_0(t,s))$ sarà denotata con $(p_5(t,x;s),\varphi_5(t;s))$. Osserviamo esplicitamente che la funzione $\varphi_5(t;\tau)$ non è continua nella coppia (t,τ), poiché non lo è la funzione $\Phi_0(t,\tau)$. Essa, comunque, appartiene a $C^{(0)}([0,T];L^1(0,T))$.

Una verifica diretta prova le seguenti proposizioni:

4.1 *Se $h(t,x)=Y_s[\gamma_1,\gamma_2,0,0](t,x)$ allora la soluzione di $S(h,0)$ è data da:*

$$p(t,x) = \sum_{i=1}^{2} \int_0^L p_i(t,x;s,\xi)\gamma_i(\xi)d\xi$$

$$\varphi(t) = \sum_{i=1}^{2} \int_0^L \varphi_i(t;s,\xi)\gamma_i(\xi)d\xi$$

4.2 *Se $h(t,x)=Y_s[0,0,v(s)+v'(s)(t-s),0](t,x)$, allora la soluzione di $S(h,0)$ è data da:*

$$\begin{aligned}
p(t,x) =& p_3(t,x;s)v(s) + p_4(t,x;s)v'(s)\\
\varphi(t,x) =& \varphi_3(t;s)v(s) + \varphi_4(t;s)v'(s)
\end{aligned}$$

4.3 *Se* $h(t,x)=Y_s[0,0,\int_s^t(t-\tau)Cz(\tau)d\tau,0](t,x)$, *e* $z(t)=\Phi(t,s)z(s)$ *allora la soluzione di* $S(h,z)$ *è data da:*

$$p(t,x)=p_5(t,x;s)z(s)\ ;\quad \varphi(t,x)=\varphi_5(t;s)z(s)$$

4.4 *Se* $h(t,x)=Y_s[0,0,\int_s^t(t-\tau)Cz(\tau)d\tau,0](t,x)$ *e* $z(t) = \int_s^t \Phi(t,\tau)B(\tau)u(\tau)d\tau$ *allora la soluzione di* $S(h,z)$ *è data da:*

$$p(t,x)=\int_s^T p_5(t,x;\tau)B(\tau)u(\tau)d\tau$$
$$\varphi(t)=\int_s^T \varphi_5(t,\tau)B(\tau)u(\tau)d\tau$$

Da tali proposizioni, ponendovi $\gamma_1(x)=y_u(s,x)$, $\gamma_2(x)=\partial_t y_u(s,x)$, si ottiene una rappresentazione per lo stato aggiunto $(p_u(t,x),\varphi_u(t))$ nel dominio D_s tramite lo stato del sistema all'istante $t=s$ e il controllo $u(t)$. Infatti (p_u,φ_u), per (1.1), (1.2), (1.3), risolve il sistema $S(y_u,z_u)$; poiché in D_s $y_u(t,x)$ è data da:

$$Y_s[y_u(s,x),\partial_t y_u(s,x),v_u(s)+v_u'(s)(t-s)+\int_s^t(t-\tau)Cz_u(\tau)d\tau,0]$$

dalle proposizioni del numero precedente traiamo allora che per $t\in[s,T]$ risulta:

$$\varphi_u(t)=\int_0^L \varphi_1(t;s,\xi)y_u(s,\xi)d\xi$$
$$+\int_0^L \varphi_2(t;s,\xi)\partial_t y_u(s,\xi)d\xi+\varphi_3(t;s)v_u(s) \tag{4.5}$$
$$+\varphi_4(t;s)v_u'(s)+\varphi_5(t;s)z_u(s)+\int_s^t \varphi_5(t;\tau)B(\tau)u(\tau)d\tau$$

5. Il feedback Sia $\tilde{u}$ il controllo ottimo; poniamo:

$$(5.1)\quad m(t;s) = \int_0^L \varphi_1(t;s,\xi) y_{\tilde{u}}(s,\xi)d\xi + \int_0^L \varphi_2(t;s,\xi)\partial_t y_{\tilde{u}}(s,\xi)d\xi$$

$$+\varphi_3(t;s)v_{\tilde{u}} + \varphi_4(t;s)v'_{\tilde{u}}(s) + \varphi_5(t;s)z_{\tilde{u}}(s)$$

$$M(t;\tau) = \varphi_5(t;\tau)B(\tau)E^{-1}(\tau)B^*(\tau)$$

e dimostriamo la proposizione:

5.1 *Se $\tilde{u}$ è il controllo ottimo, allora per ogni $s \in [0,T[$ la funzione $\varphi_{\tilde{u}}$ è l'unica soluzione dell'equazione integrale:*

$$(5.3)\qquad \varphi(t) = m(t;s) + \int_s^T M(t;\tau)\varphi(\tau)d\tau \quad t \in [s,T]$$

DIM. Dalla (4.5) e dal Teorema III si trae subito che $\varphi_{\tilde{u}}(t)$ è una soluzione dell'equazione (5.3). Essa è anche l'unica. Sia infatti $\overline{\varphi}(t)$ una soluzione di detta equazione.Poniamo:

$$\begin{aligned}
\overline{u}(t) =& E^{-1}(t)B^*(t)\overline{\varphi}(t)\\
\overline{z}(t) =& z_{\tilde{u}}(s) + \int_s^t \Phi(t,\tau)B(\tau)\overline{u}(\tau)d\tau\\
\overline{v}(t) =& v_{\tilde{u}}(s) + v'_{\tilde{u}}(s)(t-s) + \int_s^t (t-\tau)C\overline{z}(\tau)d\tau\\
\overline{h}(t,x) =& Y_s[y_{\tilde{u}}(s,x), \partial_t y_{\tilde{u}}(s,x), \overline{v}(t), 0]
\end{aligned}$$

e consideriamo il sistema $S(\overline{h},\overline{z})$. Dalle proposizioni del n.4 segue che la componente φ della soluzione di tale sistema coincide con $\overline{\varphi}$. Indichiamone con $\overline{p}(t,x)$ la prima. È immediato convincersi che il vettore:

$$\overline{X} = (\overline{z}, \overline{v}, \overline{h}, \overline{p}, \overline{\varphi})$$

è una soluzione del sistema (2.3). Dalla proposizione 2.2 segue allora, come volevamo, che $\overline{\varphi}(t) = \varphi_{\tilde{u}}(t) \quad \forall t \in [s,T]$.

Abbiamo dimostrato che l'equazione (5.3) è univocamente risolubile e che la sua soluzione appartiene a $C^{(0)}([s,T])$; inoltre, dai risultati del n.4 ricaviamo che $M(t;\tau)$ appartiene a $C^{(0)}([s,T],L^1(s,T))$. Indichiamone con $N(t,\tau;s)$ il nucleo risolvente. Dalla proposizione 5.1 traiamo che:

$$(5.4) \qquad \varphi_{\bar{u}}(t) = m(t;s) + \int_s^T N(t,\tau;s)m(\tau,s)d\tau \quad 0 \leq s \leq t \leq T$$

Indichiamo poi con $\tilde{\varphi}_i(t;s,\xi)$, $i = 1,2$ le funzioni che si ottengono ponendo, nel secondo membro della (5.4), $\varphi_i(t,s,\xi)$ al posto di m(t;s); denotiamo poi con $\tilde{\varphi}_i(t;s)$, $i = 3,4,5$ le funzioni costruite allo stesso modo a partire dalle $\varphi_i(t;s)$. Con tali notazioni la (5.4) diviene:

$$(5.5) \quad \varphi_{\bar{u}}(t) = \int_0^L \tilde{\varphi}_1(t,s;\xi)y_{\bar{u}}(s,\xi)d\xi + \int_0^L \tilde{\varphi}_2(t,s;\xi)\partial_t y_{\bar{u}}(s,\xi)d\xi$$

$$+\tilde{\varphi}_3(t;s)v_{\bar{u}}(s) + \tilde{\varphi}_4(t;s)v'_{\bar{u}}(s) + \tilde{\varphi}_5(t;s)z_{\bar{u}}(s) \quad t \in [s,T]$$

Tenendo conto della continuità dei nuclei $\tilde{\varphi}_i$ rispetto alla coppia (t,s), nel dominio $0 \leq s \leq t \leq T$ e passando al limite nella (5.6) per $s \to t^-$ si completa la dimostrazione del Teorema II.

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Su una classe di disequazioni variazionali di evoluzione del secondo ordine (*)

Nota di M. Biancardi (a Napoli) (**)

Presentata dal socio Guido Trombetti

Adunanza del 16/11/1991

Sommario: Si considera il problema di Cauchy per una classe di disequazioni variazionali di evoluzione del secondo ordine. Per esso si stabiliscono teoremi di esistenza e di regolarità.

Summary: We study the Cauchy problem for a class of second order evolutive variational inequalities. Existence and regularity theorems are proved.

Sono dati gli spazi di Hilbert reali separabili V_l, H_l $(l = 1,2)$, H e gli operatori $\gamma_l \in \mathcal{L}(V_l, H)$. Supposto $V_l \subseteq H_l$ con l'immersione densa e continua e identificato H_l al proprio duale, indichiamo con $(\cdot,\cdot)_l$, $|\cdot|_l$ [risp. $(\cdot,\cdot)$, $|\cdot|$] il prodotto scalare e la norma di H_l [risp. H], con $\|\cdot\|_l$ la norma di V_l e con $\langle\cdot,\cdot\rangle_l$ [risp. $\langle\cdot,\cdot\rangle$] la dualità tra V_l e V_l' (duale di V_l) [risp. la dualità tra $V_1 \times V_2$ e il suo duale].

Sono dati inoltre gli operatori simmetrici A_l, $B_l \in \mathcal{L}(V_l, V_l')$ per i quali si assume:

(*) Ricerca effettuata con fondi erogati dal M.U.R.S.T.

(**) Indirizzo dell'Autore: Dipartimento di Matematica e Applicazioni. Facoltà di Ingegneria, via Claudio, 21 - 80125 Napoli.

$i_1)\quad \langle A_l z, z\rangle_l \geq 0 \qquad \forall z \in V_l$.

$$i_2)\quad \begin{cases} \langle B_1 z, z\rangle_1 \geq 0 \qquad \forall z \in V_1 \; ; \\ \langle B_1 z, z\rangle_1 + \lambda_1 |z|_1^2 \geq b_1 \parallel z \parallel_1^2 \qquad \forall z \in V_1 \quad (\lambda_1 = \text{cost.} \geq 0 \; , \quad b_1 = \text{cost.} > 0) \; ; \\ \langle B_2 z, z\rangle_2 \geq b_2 \parallel z \parallel_2^2 \qquad \forall z \in V_2 \quad (b_2 = \text{cost.} > 0) \; . \end{cases}$$

Assegnati $f_l \in L^2(0,T;V_l')$, $u_{l0} \in V_l$, $u_{11} \in H_1$ ed il cono chiuso convesso **K** dello spazio H di vertice l'origine, in questo lavoro studiamo il seguente

PROBLEMA (P). Trovare $(u_1, u_2) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ $(0 < T < +\infty)$, con $u_1'' \in L^2(0,T;V_1')$, tale che:

(1) $\qquad u_l(0) = u_{l0} \, , \quad u_1'(0) = u_{11}$,

(2) $\qquad \gamma_1 u_1'(t) - \gamma_2 u_2'(t) \in \mathbf{K} \qquad$ q.o. su $]0,T[$,

(3) $$\int_0^T \left\{ \langle u_1''(t) \, , \, v_1(t) - u_1'(t)\rangle_1 + \sum_1^2{}_l \, \langle A_l u_l(t) + B_l u_l'(t) - f_l(t) \, , \, v_l(t) - u_l'(t)\rangle_l \right\} dt \geq 0$$

$$\forall (v_1, v_2) \in \prod_{l=1}^{2} L^2(0,T;V_l) \quad \text{con} \quad \gamma_1 v_1(t) - \gamma_2 v_2(t) \in \mathbf{K} \quad \text{q.o. su} \;\;]0,T[\; .$$

Con riferimento al problema (P), che ovviamente ammette al più una soluzione, dimostreremo tre teoremi di esistenza (teoremi 4, 5, 6) per i quali, oltre alle $i_1)$, $i_2)$, supporremo rispettivamente:

$i_3)$ $\quad \exists\gamma \in \mathcal{L}(V_1, V_2) : \gamma_1 z = \gamma_2 \gamma z \qquad \forall z \in V_1$.

$$i_4) \quad \begin{cases} f_1 \in L^2(0,T;H_1)\,, \; f_2 = 0\;; \\ u_{11} \in V_1\,, \quad \exists\, u_{21} \in V_2 : \gamma_1 u_{11} - \gamma_2 u_{21} \in \mathbf{K}\,. \end{cases}$$

$$i_5) \quad \begin{cases} f_1 \in H^1(0,T;V_1')\,, \; f_2 = 0\;; \\ u_{20} = 0\;; \\ u_{11} \in V_1\,, \; \gamma_1 u_{11} \in \mathbf{K}\;; \\ A_1 u_{10} + B_1 u_{11} - f_1(0) \in H_1\,. \end{cases}$$

L'acquisizione di tali risultati si basa essenzialmente sui teoremi 1, 2, 3 relativi al problema penalizzato (Problema (P_ε)) la cui dimostrazione si avvale del metodo di Faedo-Galerkin ([1], [2], [3], [4], [5], [6]) perturbando opportunamente l'equazione del problema penalizzato.

Completa il lavoro un teorema di regolarità "rispetto a x" in un caso concreto (teorema 7).

1. Denotiamo con $P_{\mathbf{K}}$ il proiettore su $\mathbf{K}$ nello spazio H e poniamo per ogni $(z_1, z_2) \in V_1 \times V_2$

$$\beta(z_1, z_2) = (\gamma_1 z_1 - \gamma_2 z_2) - P_{\mathbf{K}}(\gamma_1 z_1 - \gamma_2 z_2)\,.$$

Fissato $\varepsilon > 0$, poniamo ancora per ogni $y = (y_1, y_2)$, $z = (z_1, z_2) \in V_1 \times V_2$

$$\langle L_\varepsilon\, y, z\rangle = \frac{1}{\varepsilon}(\beta(z_1, z_2), \gamma_1 z_1 - \gamma_2 z_2)\,.$$

Rilevato che:

(4) $\beta \in C^{0,1}(V_1 \times V_2, H)$,

(5) $\beta(z_1, z_2) = \frac{1}{2}\nabla|\beta(z_1, z_2)|^2 \quad \forall (z_1, z_2) \in V_1 \times V_2$ [1] ,

(6) L_ε è limitato, monotono ed emicontinuo,

studiamo preliminarmente il

PROBLEMA (P_ε). Trovare $(u_{1\varepsilon}, u_{2\varepsilon}) \in \prod_{l=1}^{2} H^1(0,T;V_l)$, con $u''_{1\varepsilon} \in L^2(0,T;V'_1)$, in modo che:

(7) $$\langle u''_{1\varepsilon}(t), z_1\rangle_1 + \sum_1^2{}_l \, \langle A_l u_{l\varepsilon}(t) + B_l u'_{l\varepsilon}(t) - f_l(t), z_l\rangle_l +$$

$$+\frac{1}{\varepsilon}\,(\beta(u'_{1\varepsilon}(t), u'_{2\varepsilon}(t)), \gamma_1 z_1 - \gamma_2 z_2) = 0 \qquad \text{q.o. su } \;]0,T[\quad \forall (z_1, z_2) \in V_1 \times V_2 ,$$

(8) $u_{l\varepsilon}(0) = u_{l0}\,, \quad u'_{1\varepsilon}(0) = u_{11}$.

Sussiste in proposito il

TEOREMA 1. Nelle ipotesi i_1), i_2) il problema (P_ε) ammette una e una sola soluzione $(u_{1\varepsilon}, u_{2\varepsilon})$ e si ha:

(9) $$\sum_1^2{}_l \;\| u_{l\varepsilon} \|_{H^1(0,T;V_l)} \leq c \qquad (c = \text{cost.} > 0 \;\; \text{indip. da } \varepsilon) .$$

DIM. L'unicità della soluzione è ovvia. Allo scopo di provarne l'esistenza, ammettiamo, senza ledere la generalità, che $u_{11} \in V_1$ e scegliamo una base $\{z_{lj}\}$ di V_l in modo che, detto V_{ln} lo spazio generato da $\{z_{l1}, \ldots, z_{ln}\}$, si abbia $u_{l0} \in V_{l1}$ e $u_{11} \in V_{11}$. Fissato u_{21} in V_{21}, stante la (4), un noto risultato sui sistemi di equazioni differenziali ordinarie scalari assicura, per ogni $n \in N$, l'esistenza di

[1] Col simbolo ∇ indichiamo il gradiente compatto ([8], teorema 4.1, pag.303).

un unico elemento $(w_{1n}, w_{2n}) \in \prod_{l=1}^{2} H^2(0,T;V_{ln})$ talché:

$$(10) \qquad (w''_{1n}(t), z_1)_1 + \frac{1}{n}(w''_{2n}(t), z_2)_2 + \sum_1^2 {}_l \langle A_l w_{ln}(t) + B_l w'_{ln}(t) - f_l(t), z_l\rangle_l +$$

$$\frac{1}{\varepsilon}(\beta(w'_{1n}(t), w'_{2n}(t)), \gamma_1 z_1 - \gamma_2 z_2) = 0 \qquad \text{q.o. su} \quad]0,t[\quad \forall (z_1, z_2) \in V_{1n} \times V_{2n} ,$$

$$(11) \qquad w_{ln}(0) = u_{l0} , \qquad w'_{ln}(0) = u_{l1} .$$

Per la (10) con $z_l = w'_{ln}(t)$, portando in conto la (11), la i_1) e la diseguaglianza

$$(\beta(w'_{1n}(t), w'_{2n}(t)), \gamma_1 w'_{1n}(t) - \gamma_2 w'_{2n}(t)) \geq 0 \qquad \forall t \in [0,T] ,$$

intanto si ha:

$$\frac{1}{2}|w'_{1n}(t)|_1^2 + \sum_1^2 {}_l \int_0^t \langle B_l w'_{ln}(s), w'_{ln}(s)\rangle_l ds \leq$$

$$\leq \frac{1}{2}|u_{11}|_1^2 + \frac{1}{2n}|u_{21}|_2^2 + \frac{1}{2}\sum_1^2 {}_l \langle A_l u_{l0}, u_{l0}\rangle_l + \sum_1^2 {}_l \int_0^t \langle f_l(s), w'_{ln}(s)\rangle_l ds \qquad \forall t \in [0,T] .$$

D'altra parte, sussistendo la seconda e la terza delle i_2), si ha anche per ogni $\sigma > 0$:

$$\int_0^t \langle f_1(s), w'_{1n}(s)\rangle_1 ds \leq \frac{1}{2\sigma} \parallel f_1 \parallel^2_{L^2(0,T;V'_1)} + \frac{\sigma}{2b_1}\int_0^t [\lambda_1 |w'_{1n}(s)|_1^2 + \langle B_1 w'_{1n}(s), w'_{1n}(s)\rangle_1] ds ,$$

$$\int_0^t \langle f_2(s), w'_{2n}(s)\rangle_2 ds \leq \frac{1}{2\sigma} \parallel f_2 \parallel^2_{L^2(0,T;V'_2)} + \frac{\sigma}{2b_2}\int_0^t \langle B_2 w'_{2n}(s), w'_{2n}(s)\rangle_2 ds .$$

Pertanto

$$\frac{1}{2}|w'_{1n}(t)|_1^2 + \sum_1^2 {}_l \left(1 - \frac{\sigma}{2b_l}\right)\int_0^t \langle B_l w'_{ln}(s), w'_{ln}(s)\rangle_l ds \leq$$

$$\leq \frac{1}{2}\sum_{1}^{2}{}_l \left[|u_{l1}|_l^2 + \langle A_l u_{l0},\, u_{l0}\rangle_l + \frac{1}{\sigma}\ \| f_l \|_{L^2(0,T;V_l')}\right] + \frac{\lambda_1 \sigma}{2b_1}\int_0^t |w_{1n}'(s)|_1^2 ds$$

$\forall t \in [0,T]$ e $\forall \sigma > 0$,

da cui, invocando il lemma di Gronwall e le i_2), si ottiene:

(12) $\quad \| w_{1n}' \|_{C^0([0,T],H_1)} \leq c$,

$(c = \text{cost.} > 0$ indip. da ε e da $n)$

(13) $\quad \sum_{1}^{2}{}_l\ \| w_{ln} \|_{H^1(0,T;V_l)} \leq c$.

La (13) implica l'esistenza di $(u_{1\varepsilon},\, u_{2\varepsilon}) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ e di $\theta \in L^2(0,T;H)$ tali che, a meno di estratte:

(14) $\quad w_{ln} \longrightarrow u_{l\varepsilon}$ in $H^1(0,T;V_l)$ debolmente ,

(15) $\quad \beta(w_{1n}'(\cdot),\, w_{2n}'(\cdot)) \longrightarrow \theta$ in $L^2(0,T;H)$ debolmente .

Verifichiamo che $(u_{1\varepsilon},\, u_{2\varepsilon})$ è la soluzione del problema (P_ε). Anzitutto, qualunque siano $\varphi \in C_0^\infty(]0,T[)$ e $j \in N$, risulta:

$$(\int_0^T u_{1\varepsilon}'(t)\varphi'(t)dt,\, z_{1j})_1 = \langle\int_0^T [A_1 u_{1\varepsilon}(t) + B_1 u_{1\varepsilon}'(t) - f_1(t)]\varphi(t)dt,\, z_{ij}\rangle_1 +$$

$$+ (\frac{1}{\varepsilon}\int_0^T \theta(t)\varphi(t)dt,\, \gamma_1 z_{1j}) ,$$

$$\langle\int_0^T [A_2 u_{2\varepsilon}(t) + B_2 u_{2\varepsilon}'(t) - f_2(t)]\varphi(t)dt,\, z_{2j}\rangle_2 = (\frac{1}{\varepsilon}\int_0^T \theta(t)\varphi(t)dt,\, \gamma_2 z_{2j}) ,$$

grazie alle (10), (14), (15). Valendo l'uguaglianza

$$\overline{\bigcup_{n\in N} V_{ln}} = V_l ,$$

si ha allora:

$$u''_{1\varepsilon} \in L^2(0,T;V'_1)$$

$$(16) \qquad \langle u''_{1\varepsilon}(t),\, z_1\rangle_1 + \sum_1^2{}_l\ \langle A_l u_{l\varepsilon}(t) + B_l u'_{l\varepsilon}(t) - f_l(t)\,,\, z_l\rangle_l +$$

$$+\frac{1}{\varepsilon}(\theta(t),\, \gamma_1 z_1 - \gamma_2 z_2) = 0 \qquad \text{q.o. su} \quad]0,T[\quad \forall\,(z_1,\, z_2) \in V_1 \times V_2\ .$$

Aggiungiamo le convergenze:

$$(17) \qquad w_{ln}(t) \longrightarrow u_{l\varepsilon}(t) \quad \text{in} \quad V_l \quad \text{debolmente} \qquad \forall\, t \in [0,T]\ ,$$

$$(18) \qquad w'_{1n}(t) \longrightarrow u'_{1\varepsilon}(t) \quad \text{in} \quad H_1 \quad \text{debolmente} \qquad \forall\, t \in [0,T]\ .$$

La (17) è causata dalla (14). Per quanto concerne la (18), sussistendo la (12) e la densità di $\{z_{1j}\}$ in H_1, basta accertare che

$$(19) \qquad \lim_n\, (w'_{1n}(t),\, z_{1j})_1 = (u'_{1\varepsilon}(t),\, z_{1j})_1 \qquad \forall\, t \in [0,T]\ .$$

Sia $t_0 \in [0,T[$. Scelta $\psi \in C^1([0,T])$ con $\psi(t_0) = 1$ e $\psi(T) = 0$, poiché per la (10):

$$(w'_{1n}(t_0),\, z_{1j})_1 = (-\int_{t_0}^T w''_{1n}(t)\psi(t)\, dt,\, z_{1j})_1 + (-\int_{t_0}^T w'_{1n}(t)\psi'(t)dt,\, z_{1j})_1 =$$
$$= \int_{t_0}^T \langle A_1 w_{1n}(t) + B_1 w'_{1n}(t) - f_1(t),\, \psi(t) z_{1j}\rangle_1 dt + \frac{1}{\varepsilon}\int_{t_0}^T (\beta(w'_{1n}(t),\, w'_{2n}(t)),\, \psi(t)\gamma_1 z_{1j}) dt +$$
$$- \int_{t_0}^T (w'_{1n}(t),\, \psi'(t) z_{1j})_1 dt \qquad \forall\, n \geq j$$

sfruttando le (14), (15), (16), si perviene alla (19) con $t = t_0$. Se $t_0 = T$, si procede come sopra scegliendo $\psi \in C^1([0,T])$ in modo che $\psi(0) = 0$ e $\psi(T) = 1$.

Le (11), (17), (18) ci dicono che $(u_{1\varepsilon}, u_{2\varepsilon})$ soddisfa alle (8). Disponendo della (16), vale anche la (7) purché

$$\beta(u'_{1\varepsilon}(\cdot),\, u'_{2\varepsilon}(\cdot)) = \theta$$

e questa, alla luce delle (14), (15), è senz'altro vera se

$$(20)\quad \lim_n{}'' \frac{1}{\varepsilon}\int_0^T (\beta(w'_{1n}(t),\, w'_{2n}(t)),\, \gamma_1 w'_{1n}(t) - \gamma_2 w'_{2n}(t))dt \le \frac{1}{\varepsilon}\int_0^T (\theta(t),\, \gamma_1 u'_{1\varepsilon}(t) - \gamma_2 u'_{2\varepsilon}(t))d$$

in quanto per la (6) l'operatore

$$v \in L^2(0,T;V_1\times V_2) \longrightarrow L_\varepsilon v(\cdot)$$

è limitato, monotono ed emicontinuo ([2], proporzione 2.5, pag.179).

La (10), ove si ponga $z_l = w'_{ln}(t)$, e le (11) consentono la diseguaglianza

$$\frac{1}{\varepsilon}\int_0^T (\beta(w'_{1n}(t),\, w'_{2n}(t)),\, \gamma_1 w'_{1n}(t) - \gamma_2 w'_{2n}(t))dt \le$$
$$\sum_1^2 {}_l\int_0^T \langle f_l(t),\, w'_{2n}(t)\rangle_l dt + \frac{1}{2}|u_{11}|_1^2 + \frac{1}{2n}|u_{21}|_2^2 + \frac{1}{2}\sum_1^2 {}_l\langle A_l u_{l0},\, u_{l0}\rangle_l +$$
$$-\left\{\frac{1}{2}|w'_{1n}(T)|^2 + \sum_1^2 {}_l\left[\frac{1}{2}\langle A_l w_{ln}(T),\, w_{ln}(T)\rangle_l + \int_0^T \langle B_l w'_{ln}(t),\, w'_{ln}(t)\rangle dt\right]\right\}$$

da cui, con l'intervento delle (14), (17), (18), si ricava:

$$\lim_n{}'' \frac{1}{\varepsilon}\int_0^T (\beta(w'_{1n}(t),\, w'_{2n}(t)),\, \gamma_1 w'_{1n}(t) - \gamma_2 w'_{2n}(t))dt \le$$
$$\sum_1^2 {}_l\int_0^T \langle f_l(t),\, u'_{l\varepsilon}(t)\rangle_l dt + \frac{1}{2}|u_{11}|_1^2 + \frac{1}{2}\sum_1^2 {}_l\langle A_l u_{l0},\, u_{l0}\rangle_l +$$
$$-\left\{\frac{1}{2}|u'_{1\varepsilon}(T)|_1^2 + \sum_1^2 {}_l\left[\frac{1}{2}\langle A_l u_{l\varepsilon}(T),\, u_{l\varepsilon}(T)\rangle_l + \int_0^T \langle B_l u'_{l\varepsilon}(t),\, u'_{l\varepsilon}(t)\rangle dt\right]\right\}\ ,$$

ovvero la (20) in virtù della (16) e della già acquisita (8).

La (9), infine, è conseguenza immediata delle (13), (14).

Completiamo lo studio del problema (P_ε) con due teoremi di regolarità. Poiché la dimostrazione di entrambi è sostanzialmente simile a quella del teorema 1, ci limiteremo a segnalare soltanto quelle maggiorazioni utili al fine di conseguire il risultato desiderato.

TEOREMA 2. Nelle ipotesi $i_1)$, $i_2)$, $i_4)$ per la soluzione $(u_{1\varepsilon}, u_{2\varepsilon})$ del problema (P_ε) si ha:

$$u''_{1\varepsilon} \in L^2(0,T;H_1) \ , \quad \| u''_{1\varepsilon} \|_{L^2(0,T;H_1)} + \sum_1^2 {}_l \ \| u_{l\varepsilon} \|_{H^1(0,T;V_l)} \leq c \ .$$

$(c = \text{cost.} > 0$ indip. da $\varepsilon)$

DIM. Con riferimento alla soluzione (w_{1n}, w_{2n}) del problema (10), (11), dove u_{21} è assegnato dalla $i_4)$, oltre alle maggiorazioni (12), (13), risulta anche:

(21) $\| w''_{1n} \|_{L^2(0,T;H_1)} \leq c \ .$ $\quad (c = \text{cost.} > 0$ indip. da ε e da $n)$

Invero, tenendo presente che per la (5) e la seconda delle $i_4)$

$$\int_0^T (\beta(w'_{1n}(t), w'_{2n}(t)), \gamma_1 w''_{1n}(t) - \gamma_2 w''_{2n}(t))dt = \frac{1}{2} |\beta(w'_{1n}(T), w'_{2n}(T))|^2 \ ,$$

dalla (10), con $z_l = w''_{ln}(t)$ si ottiene:

$$\int_0^T |w''_{1n}(t)|_1^2 dt + \sum_1^2 {}_l \ b_l \ \| w'_{ln}(T) \|_l^2 \leq$$
$$\int_0^T (f_1(t), w''_{1n}(t))_1 dt + \sum_1^2 {}_l \{ \langle A_l u_{l0}, u_{l1} \rangle_l - \langle A_l w_{ln}(T), w'_{ln}(T) \rangle_l +$$
$$+ \int_0^T \langle A_l w'_{ln}(t), w'_{ln}(t) \rangle_l dt + \frac{1}{2} \langle B_l u_{l1}, u_{l1} \rangle_l \} + \lambda_1 |w'_{1n}(T)|_1^2 \ ,$$

e di qui si perviene facilmente alla (21) con l'ausilio delle (12), (13) e della disuguaglianza

$$\| w_{ln}(T) \|_l \leq \| u_{l0} \|_l + \int_0^T \| w'_{ln}(t) \|_l \ dt \ .$$

TEOREMA 3. Nelle ipotesi i_1), i_2), i_5) per la soluzione $(u_{1\varepsilon}, u_{2\varepsilon})$ del problema (P_ε) si ha:

$$u_{l\varepsilon} \in H^2(0,T;V_l) \ ; \quad u''_{1\varepsilon} \in L^\infty(0,T;H_1) \ ,$$

$$\| u''_{1\varepsilon} \|_{L^\infty(0,T;H_1)} + \sum_1^2 {}_l \ \| u_{l\varepsilon} \|_{H^2(0,T;V_l)} \leq c \qquad (c = \text{cost.} > 0 \text{ indip. da } \varepsilon)$$

DIM. Relativamente alla soluzione (w_{1n}, w_{2n}) del problema (10), (11), dove $u_{20} = u_{21} = 0$, si tratta di stabilire che

$$\| w''_{1n} \|_{C^0([0,T],H_1)} + \sum_1^2 {}_l \ \| w_{ln} \|_{H^2(0,T;V_l)} \leq c \ . \tag{22}$$

$(c = \text{cost.} > 0$ indip. da ε e da $n)$

Anzitutto

$$(w_{1n},\ w_{2n}) \in \prod_{l=1}^2 H^3(0,T;V_{ln})$$

grazie alla (4) e alla prima delle i_5). La (10), per $t = 0$, unitamente alla seconda, terza e quarta delle i_5), comporta che:

$$\begin{aligned} &|w''_{1n}(0)|_1 \leq c \ , \qquad (c = \text{cost.} > 0 \text{ indip. da } \varepsilon \text{ e da } n) \\ &w''_{2n}(0) = 0 \ . \end{aligned} \tag{23}$$

Ponendo $z_l = w''_{ln}(t)$ nella relazione che si deduce dalla (10) derivandone il primo membro e approfittando delle i_1), della seconda delle (23) e della diseguaglianza

$$\left(\frac{d}{dt}\beta(w'_{1n}(t),\ w'_{2n}(t)) \ , \ \gamma_1 w''_{1n}(t) - \gamma_2 w''_{2n}(t) \right) \geq 0 \qquad \forall t \in [0,T] \ ,$$

si ha:

$$\frac{1}{2}|w''_{1n}(t)|_1^2 + \sum_1^2 {}_l\, b_l \int_0^t \| w''_{ln}(s) \|_l^2\, ds \le \frac{1}{2}|w''_{1n}(0)|_1^2 + \frac{1}{2}\langle A_1 u_{11},\, u_{11}\rangle_1 +$$

$$+\frac{1}{2\sigma} \| f_1' \|_{L^2(0,T;V_1')} + \frac{\sigma}{2}\int_0^t \| w''_{1n}(s) \|_1^2\, ds + \lambda_1 \int_0^t |w''_{1n}(s)|_1^2 ds$$

$$\forall \sigma > 0 \qquad \forall t \in [0,T] \,.$$

Ne segue la (22) a causa della prima delle (23) e del lemma di Gronwall.

2. Siamo ora in grado di stabilire alcuni teoremi di esistenza per il problema (P).

TEOREMA 4. Nelle ipotesi $i_1)$, $i_2)$, $i_3)$ il problema (P) ammette una (e una sola) soluzione.

DIM. Sia $(u_{1\varepsilon},\, u_{2\varepsilon}) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ la soluzione del problema (P_ε), la cui esistenza è garantita dal teorema 1. Lo stesso teorema 1 fornisce la maggiorazione (9).

Poiché

$$(\beta(u'_{1\varepsilon}(t),\, u'_{2\varepsilon}(t)),\, \gamma_1 u'_{1\varepsilon}(t) - \gamma_2 u'_{2\varepsilon}(t)) = |\beta(u'_{1\varepsilon}(t),\, u'_{2\varepsilon}(t))|^2 \,,$$

usufruendo della (7) con $z_l = u'_{l\varepsilon}(t)$ e delle (8), (9), si ha:

$$\text{(24)} \qquad \frac{1}{\varepsilon}\int_0^T |\beta(u'_{1\varepsilon}(t),\, u'_{2\varepsilon}(t))|^2 dt \le c \,. \qquad (c = \text{cost.} > 0 \ \text{ indip. da } \varepsilon)$$

Grazie alla $i_3)$, ponendo nella (7), per ogni $v \in L^2(0,T;V_1)$, $z_1 = v(t)$ e $z_2 = \gamma v(t)$, si constata, per effetto della (9), che

$$\text{(25)} \qquad \| u''_{1\varepsilon} \|_{L^2(0,T;V_1')} \le c \,. \qquad (c = \text{cost.} > 0 \ \text{ indip. da } \varepsilon)$$

Consequenziale delle (9), (25) è l'esistenza di $(u_1, u_2) \in \prod_{l=1}^{2} H^1(0,T;V_l)$, con $u_1'' \in L^2(0,T;V_1')$, e di una successione infinitesima $\{\varepsilon_n\}$ di numeri positivi in guisa che, per $n \to +\infty$:

$$
(26) \qquad \begin{array}{llll}
u_{l\varepsilon_n} \longrightarrow u_l & \text{in} \quad H^1(0,T;V_l) & \text{debolmente} \; , \\
u''_{1\varepsilon_n} \longrightarrow u''_1 & \text{in} \quad L^2(0,T;V_1') & \text{debolmente} \; ,
\end{array}
$$

nonché

$$
(27) \qquad \begin{array}{llll}
u_{l\varepsilon_n}(t) \longrightarrow u_l(t) & \text{in} \quad V_l & \text{debolmente} & \forall t \in [0,T] \; , \\
u'_{1\varepsilon_n}(t) \longrightarrow u'_1(t) & \text{in} \quad H_1 & \text{debolmente} & \forall t \in [0,T] \; .
\end{array}
$$

Controlliamo che (u_1, u_2) è la soluzione del problema (P).

Le (8), (27) conducono alle (1). La (24), coadiuvata dalla prima delle (26), dà luogo alla (2), giacché ([8], lemma 1.5, pag.245)

$$
\| \beta(u_1'(\cdot), u_2'(\cdot)) \|_{L^2(0,T;H)} \leq \liminf_n{}' \; \| \beta(u'_{1\varepsilon_n}(\cdot), u'_{2\varepsilon_n}(\cdot)) \|_{L^2(0,T;H)} \; .
$$

Infine, per ogni $(v_1, v_2) \in \prod_{l=1}^{2} L^2(0,T;V_l)$ con $\gamma_1 v_1(t) - \gamma_2 v_2(t) \in \mathbf{K}$ q.o. su $]0,T[$, osservato che

$$
(\beta(u'_{1\varepsilon_n}(t), u'_{2\varepsilon_n}(t)), \gamma_1(v_1(t) - u'_{1\varepsilon_n}(t)) - \gamma_2(v_2(t) - u'_{2\varepsilon_n}(t))) =
$$

$$
= (\gamma_1 u'_{1\varepsilon_n}(t) - \gamma_2 u'_{2\varepsilon_n}(t) - P_{\mathbf{K}}(\gamma_1 u'_{1\varepsilon_n}(t) - \gamma_2 u'_{2\varepsilon_n}(t)), \gamma_1 v_1(t) - \gamma_2 v_2(t) - P_{\mathbf{K}}(\gamma_1 u'_{1\varepsilon_n}(t) - \gamma_2 u'_{2\varepsilon_n}(t))) +
$$

$$
-|\beta(u'_{1\varepsilon_n}(t), u'_{2\varepsilon_n}(t))|^2 \leq 0 \qquad \text{q.o. su} \quad]0,T[\; ,
$$

sfruttando la (7), con $z_l = v_l(t) - u'_{l\varepsilon_n}(t)$, si ha:

$$\int_0^T \langle u''_{1\varepsilon_n}(t),\, v_1(t)\rangle_1 dt + \sum_1^2{}_l \{\langle A_l u_{l\varepsilon_n}(t) + B_l u'_{l\varepsilon_n}(t),\, v_l(t)\rangle_l - \langle f_l(t),\, v_l(t) - u'_{l\varepsilon_n}(t)\rangle_l\} dt \geq$$

$$\geq \frac{1}{2}|u'_{1\varepsilon_n}(T)|_1^2 - \frac{1}{2}|u_{11}|_1^2 + \sum_1^2{}_l \left\{\frac{1}{2}\langle A_l u_{l\varepsilon_n}(T),\, u_{l\varepsilon_n}(T)\rangle_l - \frac{1}{2}\langle A_l u_{l0},\, u_{l0}\rangle_l + \right.$$

$$\left. + \int_0^T \langle B_l u'_{l\varepsilon_n}(t),\, u'_{l\varepsilon_n}(t)\rangle_l dt\right\}$$

e di qui la (3), in virtù delle (26), (27).

Lo stesso procedimento seguito per il teorema 4, col supporto dei teoremi 2 e 3, ci permette di stabilire i teoremi seguenti.

TEOREMA 5. Nelle ipotesi i_1), i_2), i_4) il problema (P) ammette una (e una sola) soluzione $(u_1,\, u_2)$ e si ha:

$$u''_1 \in L^2(0, T; H_1)\ .$$

TEOREMA 6. Nelle ipotesi i_1), i_2), i_5) il problema (P) ammette una (e una sola) soluzione $(u_1,\, u_2)$ e si ha

$$u_l \in H^2(0, T; V_l)\ , \qquad u''_1 \in L^\infty(0, T; H_1)\ .$$

3. Esemplifichiamo il problema (P) evidenziando due situazioni in cui le condizioni richieste dai teoremi del n.2 sono facilmente realizzabili.

Denotiamo con Ω_1 ed Ω_2 due aperti di R^n limitati e connessi: Ω_1 di classe C^0, $\Omega_2 \subseteq \Omega_1$.

Scegliamo V_1 e V_2 in modo che

$$H_0^2(\Omega_1) \subseteq V_1 \subseteq H^2(\Omega_1)\ , \qquad H_0^1(\Omega_2) \subseteq V_2 \subseteq H^1(\Omega_2)$$

e assumiamo

$$H_l = L^2(\Omega_l) \ , \qquad H = L^2(\Omega_2) \quad [\text{risp. } H = L^2(\partial\Omega_2) \text{ se } \Omega_2 \text{ è di classe } C^{0,1}] \ ,$$

$$\mathbf{K} = \{z \in H : z \leq 0\} \ .$$

Evidentemente:

$$\forall z \in H \qquad z - P_{\mathbf{K}} z = z^+ \ .$$

Indicato con ω l'operatore di restrizione a Ω_2 definito in V_1 e con γ_0 l'operatore identico di $H_1(\Omega_2)$ in sé [risp. l'operatore traccia di ordine zero su $\partial\Omega_2$ definito in $H^1(\Omega_2)$], poniamo:

$$\gamma_1 = \gamma_0 \circ \omega \ , \qquad \gamma_2 = \gamma_0 \ .$$

Assumiamo come operatori A_l e B_l quelli così definiti:

$$\langle A_l y, z\rangle_l = (k_l y, z)_l \quad \forall y, z \in V_l \ , \quad k_l \in L^\infty(\Omega_l) \quad \text{e} \quad k_l \geq 0 \ ;$$

$$\langle B_1 y, z\rangle_1 = \sum_{\substack{|r|=2 \\ |s|=2}} \int_{\Omega_1} a_{rs} D^r y D^s z \, dx \qquad \forall y, z \in V_1 \ , \qquad a_{rs} = a_{sr} \in L^\infty(\Omega_1) \ ;$$

$$\langle B_2 y, z\rangle_2 = \sum_1^n {}_{ij} \int_{\Omega_2} b_{ij} y_{x_i} z_{x_j} dx + \int_{\Omega_2} byz \, dx \qquad \forall y, z \in V_2 \ ,$$

$$b_{ij} = b_{ji} \ , \quad b \in L^\infty(\Omega_2) \quad \text{e} \quad b \geq b_0 \quad \text{con} \quad b_0 = \text{cost.} > 0 \ .$$

Ammettiamo che:

$$\langle B_1 z, z\rangle_1 \geq b' \sum_{|r|=2} \int_{\Omega_1} |D^r z|^2 dx \qquad \forall z \in V_1 \ , \qquad (b' = \text{cost.} > 0)$$

$$\sum_1^n {}_{ij} \int_{\Omega_2} b_{ij} z_{x_i} z_{x_j} \, dx \geq b'' \sum_1^n {}_i \int_{\Omega_2} |z_{x_i}|^2 dx \qquad \forall z \in V_2 \ . \qquad (b'' = \text{cost.} > 0)$$

Poiché Ω_1 è di classe C^0, la prima di tali disuguaglianze implica che [5]

$$\langle B_1 z, z\rangle_1 + |z|_1^2 \geq b_1 \parallel z \parallel_1^2 \qquad \forall z \in V_1 \ . \qquad (b_1 = \text{cost.} > 0)$$

Spazi, operatori e convesso verificano le ipotesi precisate nella introduzione. Se $\omega(V_1) \subseteq V_2$ valgono la $i_3)$ con $\gamma = \omega$ e la seconda delle $i_4)$ con $u_{21} = \omega(u_{11})$; se $u_{11} \leq 0$ su Ω_2 sussiste anche la seconda delle $i_6)$. Nel caso particolare:

$$\Omega_1 = \Omega_2 = \Omega \ , \quad V_1 = H_0^2(\Omega) \ , \quad V_2 = H_0^1(\Omega) \ , \quad H = L^2(\Omega) \ ,$$

sussiste il

TEOREMA 7. Nelle ipotesi:

$$\Omega \text{ di classe } C^{3,1} \ , \quad a_{rs} \in C^{1,1}(\bar{\Omega}) \ , \quad b_{ij} \in C^{0,1}(\bar{\Omega}) \ ,$$

$$u_{10} \in H_0^2(\Omega) \cap H^4(\Omega) \ , \quad u_{20} \in H_0^1(\Omega) \cap H^2(\Omega) \ , \quad u_{11} \in H_0^2(\Omega) \ ,$$

$$f_1 \in L^2(0,T;L^2(\Omega)) \ , \quad f_2 = 0 \ ,$$

per la soluzione $(u_1,\, u_2)$ del problema (P) si ha:

$$u_1 \in H^1(0,T;H^4(\Omega)) \ , \quad u_2 \in H^1(0,T;H^2(\Omega)) \ .$$

DIM. Sia $(u_{1\varepsilon},\, u_{2\varepsilon}) \in H^1(0,T;H_0^2(\Omega)) \times H^1(0,T;H_0^1(\Omega))$ la soluzione del problema (P_ε). Il teorema 2 garantisce che $u''_{1\varepsilon} \in L^2(0,T;L^2(\Omega))$ e che

(28) $$\| u''_{1\varepsilon} \|_{L^2(0,T;L^2(\Omega))} + \| u_{1\varepsilon} \|_{H^1(0,T;H_0^2(\Omega))} + \| u_{2\varepsilon} \|_{H^1(0,T;H_0^1(\Omega))} \leq c \ .$$

$$(c = \text{cost.} > 0 \ \text{ indip. da } \varepsilon)$$

Poiché la (7) equivale alle relazioni:

$$
(29) \qquad
\begin{aligned}
\langle B_1 u'_{1\varepsilon}(t),\, z_1\rangle_1 &= \left(f_1(t) - u''_{1\varepsilon}(t) - k_1 u_{1\varepsilon}(t),\, z_1\right) - \frac{1}{\varepsilon}\left([u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+,\, z_1\right) \\
&\quad \text{q.o. su } \;]0,T[\quad \forall z_1 \in H_0^2(\Omega)\ , \\
\langle B_2 u'_{2\varepsilon}(t),\, z_2\rangle_2 &= \frac{1}{\varepsilon}([u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+,\, z_2) - (k_2 u_{2\varepsilon}(t),\, z_2) \\
&\quad \text{q.o. su } \;]0,T[\quad \forall z_2 \in H_0^1(\Omega)\ ,
\end{aligned}
$$

sulla scorta di un noto risultato di regolarità ellittica [5] si ha:

$$
u'_{1\varepsilon} \in L^2(0,T;H^4(\Omega))\ , \qquad u'_{2\varepsilon} \in L^2(0,T;H^2(\Omega))\ ,
$$

$$
(30) \qquad
\begin{aligned}
\| u'_{1\varepsilon} \|_{L^2(0,T;H^4(\Omega))} \leq\ & c\Big[\| f_1 - u''_{1\varepsilon} - k_1 u_{1\varepsilon} \|_{L^2(0,T;L^2(\Omega))} + \\
&+\frac{1}{\varepsilon} \| [u'_{1\varepsilon} - u'_{2\varepsilon}]^+ \|_{L^2(0,T;L^2(\Omega))}\Big]\ , \\
& \qquad\qquad (c = \text{cost.} > 0 \text{ indip. da } \varepsilon) \\
\| u'_{2\varepsilon} \|_{L^2(0,T;H^2(\Omega))} \leq\ & c\Big[\| k_2 \|_{L^\infty(\Omega)} \cdot \| u_{2\varepsilon} \|_{L^2(0,T;L^2(\Omega))} + \\
&+\frac{1}{\varepsilon} \| [u'_{1\varepsilon} - u'_{2\varepsilon}]^+ \|_{L^2(0,T;L^2(\Omega))}\Big]\ .
\end{aligned}
$$

Rilevato che

$$
\int_0^T \langle B_2 u'_{2\varepsilon}(t) - B_2 u'_{1\varepsilon}(t)\,,\; [u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+\rangle_2 dt \leq 0\ ,
$$

avvalendosi della (29) con $z_2 = [u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+$, si ottiene:

$$(31) \qquad \frac{1}{\varepsilon}\int_0^T |[u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+|^2\, dt \le \int_0^T \langle B_2 u'_{1\varepsilon}(t)\,,\ [u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+\rangle_2 dt +$$

$$+ \int_0^T (k_2 u_{2\varepsilon}(t),\, [u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t)]^+) dt \ \le$$

$$\le [\| B_2 u'_{1\varepsilon} \|_{L^2(0,T;L^2(\Omega))} + \| k_2 \|_{L^\infty(\Omega)} \cdot \ \| u_{2\varepsilon} \|_{L^2(0,T;L^2(\Omega))}] \cdot \| [u'_{1\varepsilon} - u'_{2\varepsilon}]^+ \|_{L^2(0,T;L^2(\Omega))}$$

Dalle (28), (30), (31) si trae che

$$\| u'_{1\varepsilon} \|_{L^2(0,T;H^4(\Omega))} \le c\,, \qquad \qquad (c = \text{cost.} > 0 \text{ indip. da}\, \varepsilon)$$
$$\| u'_{2\varepsilon} \|_{L^2(0,T;H^2(\Omega))} \le c\,,$$

e tanto basta per l'asserto.

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SINGULAR PERTURBATIONS WITH MOVING BOUNDARY

Nota di Berardino D'Acunto *
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Adunanza del 16.11.1991

Riassunto Si considera un problema di perturbazione singolare in domini generali per l'*equazione dei telegrafisti non omogenea* con un piccolo parametro e l'*equazione del calore non omogenea*. Si assegnano dati arbitrari sulla frontiera mobile insieme con le condizioni iniziali. Si provano rigorose formule di approssimazioni e la uniforme convergenza delle soluzioni.

Abstract We consider a singular perturbations problem in general regions for the *inhomogeneous telegraphist equation* with a small parameter and the *inhomogeneous heat equation*. We give arbitrary data on a general moving boundary together with the initial conditions. We show rigorous approximations formulas and the uniform convergence of the solutions.

1 Introduction

In a famous paper [2] Cattaneo introduced a hyperbolic equation in order to remove the paradoxe of the infinite speed of the thermals waves connected with the *heat equation* of the classical Fourier theory. Basing his arguments on statistical mechanics, Cattaneo derived a *hyperbolic heat equation* with an inertial small coefficient ε denoted *material relaxation time*.

Some years later Zlamal [13, 14] and Fulks and Guenther [8] studied the approximation of the solutions of the hyperbolic and parabolic heat equations with reference to the Cauchy and the initial-boundary values problems. However, to my knowledge, singular perturbations problems with moving boundary have never been considered.

Recently, an increasing interest has been addressed to hyperbolic models for the study of melting problems, particularly when high energies are involved [5]. Thus, several Authors have considered hyperbolic free boundary problems with reference to heat phase changes [7, 9, 12, 3].

In connection with these questions in this paper we want to discuss the hyperbolic-parabolic singular perturbations related to the *inhomogeneous telegraphist*

*Dipartimento di Matematica e Applicazioni, Facoltà di Ingegneria, via Claudio 21, 80125, Napoli.

equation and the *inhomogeneous heat equation*. We consider an initial-boundary values problem with arbitrary data prescribed on the moving boundary $x = r(t)$. In this way we generalize also some results obtained in [4].

According to the modern formulations of the singular perturbations problems (see e.g. [6, 10, 11]), we deduce rigorous and explicit estimates and prove the uniform convergence of the solutions. Precisely, denoting by $u_\varepsilon(x,t)$ and $u(x,t)$ the solutions of the two problems we get $|u_\varepsilon - u| \leq k\varepsilon^p$, with k constant independent of ε, x, t and $p > 0$.

The singular perturbations problem we examine is different from the usual ones mainly for the presence of the moving boundary and for the form of the solutions that can be given only by means of integral equations. So, after having defined the problem in Section 2, we prove some necessary estimates involving also the moving boundary in Section 3. Finally, in Section 4 we show rigorous approximation formulas for Volterra integral equations in order to obtain the uniform convergence as well as a precise estimate of the solutions.

2 Statement of the problem

In this paper we denote by α the thermal diffusivity and by ε the material relaxation time. Moreover, we indicate by $x = r(t)$ the equation of the moving boundary. We want to study the convergence of the solution of a boundary value problem for the inhomogeneous telegraphist equation to the solution of the corresponding problem for the heat equation.

The hyperbolic problem is the following

$$(2.1) \qquad (\varepsilon\partial_t^2 - \alpha\partial_x^2 + \partial_t)u_\varepsilon(x,t) = f(x,t), \; x > r(t), 0 < t < T,$$

$$(2.2) \qquad u_\varepsilon(x,0) = \phi(x), \; u_{\varepsilon,t}(x,0) = \psi(x),$$

$$(2.3) \qquad u_\varepsilon(r(t),t) = a(t).$$

In the parabolic case, we have, obviously, only an initial condition

$$(2.4) \qquad (-\alpha\partial_x^2 + \partial_t)u(x,t) = f(x,t), x > r(t), 0 < t < T,$$

$$(2.5) \qquad u(x,0) = \phi(x),$$

$$(2.6) \qquad u(r(t),t) = a(t).$$

We consider the singular perturbations problem with moving boundary on the region

$$(2.7) \qquad \Omega = \{(x_0, t_0) \; : \; 0 < t_0 < T, \; r(t_0) < x_0 < t_0\sqrt{\alpha/\varepsilon}\}, T > 0.$$

However, for $x_0 > t_0\sqrt{\varepsilon/\alpha}$ the hyperbolic boundary values problem becames a Cauchy problem and the solution converges to the corresponding solution of the parabolic initial values problem [8]. On the data we assume that

$$\varphi \in C^2([0,+\infty[),\ \varphi(0) = a(0), \tag{2.8}$$

$$|\varphi(x)| < M_\varphi,\ |\varphi'(x)| < M'_\varphi,\ |\varphi''(x)| < M''_\varphi, \tag{2.9}$$

$$\psi \in C^1([0, 2T\sqrt{\alpha/\varepsilon}]),\ |\psi(x)| < M_\psi,\ |\psi'(x)| < M'_\psi, \tag{2.10}$$

$$a \in C^2([0,T]),\ |a(t)| < M_a,\ |\dot a(t)| < M'_a,\ |\ddot a(t)| < M''_a, \tag{2.11}$$

$$f \in C^1(\{(x,t)\ :\ 0 < t < T, r(t) < x < \infty\}), \tag{2.12}$$

$$|f(x,t)| < M_f,\ |f_x(x,t)| < M'_f. \tag{2.13}$$

In (2.9)-(2.13) M_φ, M'_φ, M''_φ, M_ψ, M'_ψ, M_a, M'_a, M''_a, M_f, M'_f are positive constants. Moreover, on the function $r(t)$ describing the moving boundary, we assume

$$r(t) \in C^2([0,T]),\ r(0) = 0,\ |\dot r(t)| \le r_1,\ r_1 = \text{constant} < \sqrt{\alpha/\varepsilon}. \tag{2.14}$$

Now, we introduce the *fundamental solution* of (2.1)

$$V(x_0 - x, t_0 - \tau) = \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} I_0\left(\sqrt{\frac{(t_0-\tau)^2}{4\varepsilon^2} - \frac{(x_0-x)^2}{4\alpha\varepsilon}}\right), \tag{2.15}$$

where $I_n (n \ge 0)$ is the modified Bessel function of order n and note that the solution u_ε of (2.1)-(2.3) is given by

$$\begin{aligned} u_\varepsilon(x_0,t_0) = {} & \frac{e^{-t_0/2\varepsilon}}{2}\varphi\left(x_0 + t_0\sqrt{\frac{\alpha}{\varepsilon}}\right) + \frac{a(t)}{2}e^{-\frac{t_0-t}{2\varepsilon}} + \\ & + \int_0^{x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}}[\varepsilon\psi(x) + \varphi(x)(1+\varepsilon\partial_{t_0})]V(x_0-x,t_0)dx - \\ & - \int_0^t a(\tau)[\dot r(\tau) - \varepsilon \dot r(\tau)\partial_\tau - \alpha\partial_x]V(x_0 - r(\tau), t_0-\tau)d\tau - \\ & - \int_0^t V(x_0 - r(\tau), t_0 - \tau)[\varepsilon\dot r(\tau)\dot a(\tau) + \alpha w_\varepsilon(\tau)(1 - \varepsilon\dot r^2(\tau)/\alpha)]d\tau + \\ & + \int_0^{t_0} d\tau \int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)} f(x,\tau)V(x_0-x,t_0-\tau)dx, (x_0,t_0) \in \Omega, \end{aligned} \tag{2.16}$$

where t is defined by $t = t_0 - \sqrt{\varepsilon/\alpha}\,[x_0 - r(t)]$ and $s(\tau)$ is given by

$$s(\tau) = r(\tau) \text{ for } 0 < \tau < t,\ s(\tau) = x_0 - \sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau) \text{ for } t < \tau < t_0.$$

Moreover, $w_\varepsilon(t) = u_{\varepsilon,x}(r(t),t)$ is solution of the following Volterra integral equation

$$w_\varepsilon(t)\left[1 - \dot r(t)\sqrt{\frac{\varepsilon}{\alpha}}\right] = e^{-t/2\varepsilon}\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) + 2\varepsilon\psi(0)V(r(t),t) + \tag{2.17}$$

$$+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}[\varepsilon\psi'(x)+\varphi'(x)(1+\varepsilon\partial_{t_0})]V(r(t)-x,t)dx-$$

$$-\sqrt{\frac{\varepsilon}{\alpha}}\dot{a}(t)-2\int_0^t \dot{a}(\tau)[1+\varepsilon\partial_{t_0}+\varepsilon\dot{r}(\tau)\partial_{x_0}]V(r(t)-r(\tau),t-\tau)d\tau+$$

$$+2\int_0^t[\alpha w_\varepsilon(\tau)(1-\varepsilon\dot{r}^2(\tau)/\alpha)\partial_x+f(r(\tau),\tau)]V(r(t)-r(\tau),t-\tau)d\tau+$$

$$+2\int_0^t d\tau\int_{r(\tau)}^{r(t)+\sqrt{\frac{\alpha}{\varepsilon}}(t-\tau)} f_x(x,\tau)V(r(t)-x,t-\tau)dx, 0<t<T.$$

Then, we define

$$E(x_0-x,t_0-\tau)=\frac{e^{-\frac{(x_0-x)^2}{4\alpha(t_0-\tau)}}}{\sqrt{4\pi\alpha(t_0-\tau)}} \tag{2.18}$$

and write the solution u of (2.4)-(2.6)

$$u(x_0,t_0)=\int_0^\infty \varphi(x)E(x_0-x,t_0)dx- \tag{2.19}$$

$$-\int_0^{t_0}[a(\tau)(\dot{r}(\tau)-\alpha\partial_x)E(x_0-r(\tau),t_0-\tau)+\alpha w(\tau)E(x_0-r(\tau),t_0-\tau)]d\tau+$$

$$+\int_0^{t_0}d\tau\int_{r(\tau)}^\infty f(x,\tau)E(x_0-x,t_0-\tau)dx,\ (x_0,t_0)\in\Omega,$$

with $w(t)=u_x(r(t),t)$ solution of

$$w(t)=2\int_0^\infty \varphi'(x)E(r(t)-x,t)dx-2\int_0^t \dot{a}(\tau)E(r(t)-r(\tau),t-\tau)d\tau+ \tag{2.20}$$

$$+2\int_0^t[\alpha w(\tau)\partial_x+f(r(\tau),\tau)]E(r(t)-r(\tau),t-\tau)d\tau+$$

$$+2\int_0^t d\tau\int_{r(\tau)}^\infty f_x(x,\tau)E(r(t)-x,t-\tau)dx,\ 0<t<T.$$

Finally, we want to recall two inequalities that can be easily deduced from [8, Sec.2]

$$V(x,t)\le 4E(x,t),\ \frac{e^{-\frac{t}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}}\frac{I_1\left(\sqrt{\frac{t^2}{4\varepsilon^2}-\frac{x^2}{4\alpha\varepsilon}}\right)}{\sqrt{1-\varepsilon x^2/\alpha t^2}}\le 14E(x,t),\ |x|<t\sqrt{\alpha/\varepsilon}. \tag{2.21}$$

3 Preliminary results

From (2.16), (2.19) we see that

$$u_\varepsilon(x_0,t_0)-u(x_0,t_0)=A(x_0,t_0,\varepsilon)+A_b(x_0,t_0,\varepsilon)+A_f(x_0,t_0,\varepsilon)+ \tag{3.1}$$

$$+\int_0^t \alpha\{w(\tau)-w_\varepsilon(\tau)[1-\varepsilon\dot{r}^2(\tau)/\alpha]\}V(x_0-r(\tau),t_0-\tau)d\tau,$$

where

$$(3.2)\ A(x_0,t_0,\varepsilon)=\alpha\int_0^{t_0}w(\tau)E(x_0-r(\tau),t_0-\tau)d\tau-\alpha\int_0^{t}w(\tau)V(x_0-r(\tau),t_0-\tau)d\tau,$$

$$(3.3)\qquad A_b(x_0,t_0,\varepsilon)=\frac{e^{-\frac{t_0}{2\varepsilon}}}{2}\varphi\left(x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}\right)-\int_0^{\infty}\varphi(x)E(x_0-x,t_0)dx+$$

$$+\int_0^{x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}}[\varepsilon\psi(x)+\varphi(x)(1+\varepsilon\partial_{t_0})]V(x_0-x,t_0)dx+$$

$$+\frac{e^{-\frac{t_0-t}{2\varepsilon}}}{2}a(t)-\int_0^{t_0}a(\tau)[\alpha\partial_x-\dot r(\tau)]E(x_0-r(\tau),t_0-\tau)d\tau+$$

$$+\int_0^{t}[\varepsilon a(\tau)\dot r(\tau)\partial_\tau+a(\tau)\alpha\partial_x-a(\tau)\dot r(\tau)-\varepsilon\dot r(\tau)\dot a(\tau)]V(x_0-r(\tau),t_0-\tau)d\tau,$$

$$(3.4)\qquad A_f(x_0,t_0,\varepsilon)=\int_0^{t_0}d\tau\int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)}f(x,\tau)V(x_0-x,t_0-\tau)d\tau-$$

$$-\int_0^{t_0}d\tau\int_{r(\tau)}^{\infty}f(x,\tau)E(x_0-x,t_0-\tau)d\tau.$$

From (3.1) we see that the convergence of u_ε to u is proved if we show that the right-hand member of (3.1) tends to zero with ε. We develop this analysis in the next theorems, where we give also rigorous and explicit estimates. We begin to discuss $A_f(x_0,t_0,\varepsilon)$.

Theorem 3.1 *Under the hypotheses* (2.12) – (2.14) *there exists a constant* K_f *independent of* ε, x_0, t_0, *such that*

$$(3.5)\qquad |A_f(x_0,t_0,\varepsilon)|\le K_f\varepsilon^{q_f},$$

where q_f *is a strictly positive rational number.*

Proof. By setting $X_i=x_0+(-1)^i(t_0-\tau)\sqrt{\alpha/\varepsilon},(i=1,2)$, and defining

$$(3.6)\qquad A_{f1}=\int_0^{t}d\tau\int_{r(\tau)}^{X_2}f(x,\tau)(V-E)(x_0-x,t_0-\tau)dx+$$

$$+\int_t^{t_0}d\tau\int_{X_1}^{X_2}f(x,\tau)(V-E)(x_0-x,t_0-\tau)dx,$$

$$(3.7)\qquad A_{f2}=-\int_0^{t_0}d\tau\int_{X_2}^{\infty}f(x,\tau)E(x_0-x,t_0-\tau)dx-$$

$$-\int_t^{t_0}d\tau\int_{r(\tau)}^{X_1}f(x,\tau)E(x_0-x,t_0-\tau)dx,$$

we see that

$$(3.8)\qquad A_f(x_0,t_0,\varepsilon)=A_{f1}+A_{f2}.$$

The last term is easily estimated. Indeed,

$$|A_{f2}| \leq M_f \int_0^{t_0} d\tau \int_{X_2}^{\infty} E(x_0 - x, t_0 - \tau) dx + M_f \int_t^{t_0} d\tau \int_{-\infty}^{X_1} E(x_0 - x, t_0 - \tau) dx,$$

and, therefore, by setting $z = (x - x_0)/\sqrt{4\alpha(t_0 - \tau)}$,

$$|A_{f2}| \leq \frac{2M_f}{\sqrt{\pi}} \int_0^{t_0} d\tau \int_{\sqrt{(t_0-\tau)/4\varepsilon}}^{\infty} e^{-z^2} dz \leq 8\sqrt{2} M_f \varepsilon. \tag{3.9}$$

As for A_{f1} we note that

$$|A_{f1}| \leq M_f \int_0^{t_0} d\tau \int_{X_1}^{X_2} |(V - E)(x_0 - x, t_0 - \tau)| d\tau, \tag{3.10}$$

since $X_1 < r(\tau)$ for $0 < \tau < t$. Morerover,

$$|A_{f1}| \leq A_{f11} + A_{f12}, \tag{3.11}$$

with $x_i = x_0 + (-1)^i (t_0 - \tau)\sqrt{\alpha/4\varepsilon}$, $(i = 1, 2)$,

$$A_{f11} = M_f \int_0^{t_0} d\tau \int_{x_1}^{x_2} |(V - E)(x_0 - x, t_0 - \tau)| dx, \tag{3.12}$$

$$A_{f12} = M_f \int_0^{t_0} d\tau \left(\int_{X_1}^{x_1} + \int_{x_2}^{X_2} \right) |(V - E)(x_0 - x, t_0 - \tau)| dx. \tag{3.13}$$

Considering (2.21), from (3.13) we have

$$A_{f12} \leq 5M_f \int_0^{t_0} d\tau \left\{ \int_{X_1}^{x_1} E(x_0 - x, t_0 - \tau) dx + \int_{x_2}^{X_2} E(x_0 - x, t_0 - \tau) dx \right\};$$

hence,

$$A_{f12} \leq \frac{5M_f}{\sqrt{\pi}} \int_0^{t_0} e^{-\frac{t_0 - \tau}{32\varepsilon}} d\tau \int_{-\infty}^{\infty} e^{-z^2/2} dz \leq 160\sqrt{2} M_f \varepsilon. \tag{3.14}$$

Now, we estimate A_{f11}. First, we note that, setting

$$A_{f111} = M_f \int_0^{t_0} d\tau \int_{x_1}^{x_2} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} |I_0\left(\frac{t_0 - \tau}{2\varepsilon} \sqrt{1 - \frac{\varepsilon}{\alpha} \left(\frac{x_0 - x}{t_0 - \tau} \right)^2} \right) -$$

$$- \frac{e^{\frac{t_0-\tau}{2\varepsilon}\sqrt{1 - \frac{\varepsilon}{\alpha}\left(\frac{x_0 - x}{t_0 - \tau}\right)^2}}}{\sqrt{\pi(t_0 - \tau)/\varepsilon} \left[1 - \frac{\varepsilon}{\alpha} \left(\frac{x_0 - x}{t_0 - \tau} \right)^2 \right]^{1/4}} | dx,$$

$$A_{f112} = M_f \int_0^{t_0} d\tau \int_{x_1}^{x_2} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}} e^{\frac{t_0-\tau}{2\varepsilon}\sqrt{1 - \frac{\varepsilon}{\alpha}(\frac{x_0-x}{t_0-\tau})^2}}}{\sqrt{4\pi\alpha(t_0 - \tau)}} | \frac{1}{[1 - \frac{\varepsilon}{\alpha}(\frac{x_0 - x}{t_0 - \tau})^2]^{1/4}} - 1 | dx,$$

$$A_{f113} = M_f \int_0^{t_0} d\tau \int_{x_1}^{x_2} | \frac{e^{-\frac{t_0-\tau}{2\varepsilon}} e^{\frac{t_0-\tau}{2\varepsilon}\sqrt{1 - \frac{\varepsilon}{\alpha}(\frac{x_0-x}{t_0-\tau})^2}}}{\sqrt{4\pi\alpha(t_0 - \tau)}} - E(x_0 - x, t_0 - \tau) | dx,$$

from (3.12) follows

$$A_{f11} \leq A_{f111} + A_{f112} + A_{f113}. \tag{3.15}$$

Afterwards, A_{f111} can be evaluated by applying the following property of the modified Bessel functions [15]

$$|I_n(\xi) - e^{\xi}/\sqrt{2\pi\xi}| \leq K/\xi, \ K = \text{constant}, \ \xi > 0. \tag{3.16}$$

Indeed,

$$A_{f111} \leq M_f K \int_0^{t_0} d\tau \int_{x_1}^{x_2} \frac{e^{-\frac{t_0-\tau}{2\varepsilon}}}{\sqrt{3\alpha\varepsilon}} \frac{2\varepsilon dx}{t_0-\tau} \leq \frac{4K}{\sqrt{3}} M_f \varepsilon, \tag{3.17}$$

since $1 - [\varepsilon(x_0-x)^2/\alpha(t_0-\tau)^2] > 3/4$ when $x_1 < x < x_2$. Now, we estimate A_{f112} obtaining

$$A_{f112} \leq M_f \left(\frac{4}{3}\right)^{1/4} \int_0^{t_0} d\tau \int_{x_1}^{x_2} E(x_0-x, t_0-\tau) \frac{\varepsilon(x_0-x)^2}{\alpha(t_0-\tau)^2} dx,$$

and, therefore

$$A_{f112} \leq 4M_f \left(\frac{4}{3}\right)^{1/4} \left(\frac{T\varepsilon}{\pi}\right)^{1/2} \tag{3.18}$$

Finally, we consider A_{f113} and have

$$A_{f113} \leq M_f \int_0^{t_0} d\tau \int_{x_1}^{x_2} E(x_0-x, t_0-\tau) \frac{t_0-\tau}{2\varepsilon} \left\{ 1 - \frac{\varepsilon}{2\alpha}\left(\frac{x_0-x}{t_0-\tau}\right)^2 - \right.$$

$$\left. - \sqrt{1 - \frac{\varepsilon}{\alpha}\left(\frac{x_0-x}{t_0-\tau}\right)^2} \right\} dx;$$

hence,

$$A_{f113} \leq \frac{2M_f\varepsilon}{e^2\sqrt{\pi\alpha}} \int_0^{t_0} d\tau \int_{x_1}^{x_2} \frac{dx}{(t_0-\tau)^{3/2}} \leq \frac{4M_f}{e^2}\sqrt{\frac{T\varepsilon}{\pi}}. \tag{3.19}$$

Thus, the theorem is proved.

In order to discuss $A_b(x_0, t_0, \varepsilon)$ defined by (3.3), we use (2.8) and the identities $\partial_\tau(\varepsilon \dot{a} V - \varepsilon a V_\tau + aV) + \partial_x(\alpha a V_x) = (\varepsilon \ddot{a} + \dot{a})V$, $\partial_\tau(aE) + \partial_x(\alpha a E_x) = \dot{a}E$, to rearrange it as follows

$$A_b(x_0, t_0, \varepsilon) = A_c(x_0, t_0, \varepsilon) + A_d(x_0, t_0, \varepsilon), \tag{3.20}$$

where

$$A_c(x_0, t_0, \varepsilon) = \frac{e^{-\frac{t_0}{2\varepsilon}}}{2}[\varphi(x_0 + t_0\sqrt{\alpha/\varepsilon}) - \varphi(0)] + \tag{3.21}$$

$$+ \int_0^{x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}} [\varepsilon(\psi(x) - \dot{a}(0)) + (\varphi(x) - \varphi(0))(1 + \varepsilon\partial_{t_0})] V(x_0-x, t_0) dx -$$

$$- \int_0^{\infty} [\varphi(x) - \varphi(0)] E(x_0-x, t_0) dx,$$

(3.22)
$$A_d(x_0,t_0,\varepsilon)=\int_0^{t_0}d\tau\int_{r(\tau)}^{\infty}\dot a(\tau)E(x_0-x,t_0-\tau)d\tau-$$
$$-\int_0^{t_0}d\tau\int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)}[\varepsilon\ddot a(\tau)+\dot a(\tau)]V(x_0-x,t_0-\tau)d\tau.$$

First, we study $A_c(x_0,t_0,\varepsilon)$.

Theorem 3.2 *Under the hypotheses* $(2.8)-(2.11),(2.14)$ *there exist a constant* K_c *independent of* ε, x_0, t_0 *and a strictly positive rational number* q_c *such that*

$$|A_c(x_0,t_0,\varepsilon)|\le K_c\varepsilon^{q_c}.$$

Proof. From (3.21) we have $A_c(x_0,t_0,\varepsilon)=A_{c1}+A_{c2}$, where

$$A_{c1}=\int_0^{Y_2}[\varphi(x)-\varphi(0)](1+\varepsilon\partial_{t_0})V(x_0-x,t_0)dx-\int_0^{\infty}[\varphi(x)-\varphi(0)]E(x_0-x,t_0)dx,$$

$$A_{c2}=\frac{e^{-\frac{t_0}{2\varepsilon}}}{2}\left[\varphi\left(x_0+t_0\sqrt{\frac{\alpha}{\varepsilon}}\right)-\varphi(0)\right]+\int_0^{Y_2}\varepsilon[\psi(x)-\dot a(0)]V(x_0-x,t_0)dx,$$

and $Y_i=x_0+(-1)^i t_0\sqrt{\alpha/\varepsilon},(i=1,2)$. This last term can be estimated as follows

$$|A_{c2}|\le M'_\varphi\frac{e^{-\frac{t_0}{2\varepsilon}}}{2}(x_0+t_0\sqrt{\alpha/\varepsilon})+4\varepsilon(M_\psi+M'_a)\int_{Y_1}^{Y_2}E(x_0-x,t_0)dx,$$

and, therefore,

$$|A_{c2}|\le 2M'_\varphi\sqrt{\alpha\varepsilon}+4(M_\psi+M'_a)\varepsilon.$$

As to A_{c1} we obtain

$$|A_{c1}|\le 10\int_0^{\infty}|\varphi(x)-\varphi(0)|E(x_0-x,t_0)dx,$$

since $(1+\varepsilon\partial_{t_0})V(x_0-x,t_0)\le 9E(x_0-x,t_0)$ for (2.21). Thus, by assuming $t_0<\varepsilon^{3/4}$ and introducing $y=(x-x_0)/\sqrt{4\alpha t_0}$, we have

$$|A_{c1}|\le\frac{5M'_\varphi}{\sqrt{\pi\alpha t_0}}\int_0^{x_0+\sqrt{t_0\alpha/\varepsilon^{1/4}}}x\,dx+\frac{20M_\varphi}{\sqrt{\pi}}\int_{1/2\varepsilon^{1/8}}^{\infty}e^{-y^2}dy.$$

Hence,

$$|A_{c1}|\le 10\sqrt{\frac{\alpha}{\pi}}M'_\varphi\varepsilon^{1/8}+80\sqrt{2}M_\varphi\varepsilon^{1/4}.$$

If, instead, $t_0\ge\varepsilon^{3/4}$, by using (2.21) we have

$$|A_{c1}|\le A_{c11}+A_{c12},$$

where

$$A_{c11}=2M_\varphi\int_{y_1}^{y_2}|(1+\varepsilon\partial_{t_0})V(x_0-x,t_0)-E(x_0-x,t_0)|dx,$$

$$A_{c12} = 20M_\varphi \int_{-\infty}^{y_1} E(x_0 - x, t_0)dx + 22M_\varphi \int_{y_2}^{\infty} E(x_0 - x, t_0)dx,$$

with $y_i = x_0 + (-1)^i t_0\sqrt{\alpha/4\varepsilon}, (1 = 1, 2)$. For A_{c12} we easily obtain

$$A_{c12} \le \frac{42M_\varphi}{\sqrt{\pi}} e^{-t_0/32\varepsilon} \int_{\sqrt{t_0/16\varepsilon}}^{\infty} e^{-y^2/2}dy \le 672\sqrt{2}M_\varphi \varepsilon^{1/4}.$$

Moreover, we can estimatate A_{c11} by the same arguments used for A_{f11} in Th.3.1 and get

$$A_{c11} \le \frac{4}{\sqrt{\pi}} M_\varphi \left[\sqrt{\frac{\pi}{e}}\frac{K}{3} + \left(\frac{4}{3}\right)^{3/4} + \frac{1}{e^2}\right] \varepsilon^{1/8}.$$

So, the theorem is completely proved.

Next, we discuss A_d.

Theorem 3.3 *Under the hypotheses* (2.8) – (2.11), (2.14) *there exists a constant* K_d *independent of* ε, x_0, t_0 *such that* $|A_d(x_0, t_0, \varepsilon)| \le K_d \varepsilon^{q_d}$, *where* q_d *is a strictly positive rational number.*

Proof. Defining

$$A_{d1} = -\int_0^{t_0} \dot{a}(\tau)d\tau \int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)} V(x_0 - x, t_0 - \tau)dx +$$

$$+\int_0^{t_0} \dot{a}(\tau)d\tau \int_{r(\tau)}^{\infty} E(x_0 - x, t_0 - \tau)dx,$$

$$A_{d2} = -\varepsilon \int_0^{t_0} \ddot{a}(\tau)d\tau \int_{s(\tau)}^{x_0+\sqrt{\frac{\alpha}{\varepsilon}}(t_0-\tau)} V(x_0 - x, t_0 - \tau)dx,$$

from (3.22) we have

$$A_d = A_{d1} + A_{d2}.$$

Moreover,

$$|A_{d2}| \le \frac{4M_a''\varepsilon}{\sqrt{\pi}} \int_0^{t_0} d\tau \int_{-\sqrt{(t_0-\tau)/4\varepsilon}}^{\sqrt{(t_0-\tau)/4\varepsilon}} e^{-z^2}dz \le 4M_a''T\varepsilon.$$

On the other hand, A_{d1} can be treated as A_f in Th.3.1.

Finally, we give an estimate on $A(x_0, t_0, \varepsilon)$.

Theorem 3.4 *Under the hypotheses* (2.8) – (2.14) *there exists a constant* K_A *independent of* ε, x_0, t_0 *such that*

$$A(x_0, t_0, \varepsilon)| \le K_A \varepsilon^{q_A},$$

where q_A *is a strictly positive rational number.*

Proof. It can be obtained by [3, Th.5.1] with obvious modifications.

Recalling Theorems 3.1-3.4 and formula (3.20) we see that the proof of the convergence is completed if we show a similar theorem for the last term of the right-hand side of (3.1). However, this requires a preliminary analysis involving integral equations that we develop in the next section.

4 Main theorem

From (2.17) and (2.20) we deduce

$$(4.1)\qquad w_\varepsilon(t)[1-\varepsilon\dot r^2(t)/\alpha]-w(t)=B(t,\varepsilon)+B_b(t,\varepsilon)+B_c(t,\varepsilon)+B_f(t,\varepsilon)+$$

$$+B_F(t,\varepsilon)\dot r(t)\sqrt{\frac{\varepsilon}{\alpha}}+\int_0^t H(t,\tau,\varepsilon)\{w_\varepsilon(\tau)[1-\varepsilon\dot r^2(\tau)/\alpha]-w(\tau)\}d\tau,$$

where we have used the following

$$(4.2)\qquad H(t,\tau,\varepsilon)=2\alpha V_x(r(t)-r(\tau),t-\tau),$$

$$(4.3)\qquad B(t,\varepsilon)=2\alpha\int_0^t w(\tau)(V_x-E_x)(r(t)-r(\tau),t-\tau)d\tau,$$

$$(4.4)\qquad B_b(t,\varepsilon)=2\int_0^t \dot a(\tau)(E-V)(r(t)-r(\tau),t-\tau)d\tau-$$

$$-\dot a(t)\sqrt{\frac{\varepsilon}{\alpha}}-2\int_0^t \dot a(\tau)\varepsilon(\partial_{t_0}+\dot r(\tau)\partial_{x_0})V(r(t)-r(\tau),t-\tau)d\tau,$$

$$(4.5)\qquad B_c(t,\varepsilon)=e^{-t/2\varepsilon}\varphi'(r(t)+t\sqrt{\alpha/\varepsilon})-2\int_0^\infty \varphi'(x)E(r(t)-x,t)dx+$$

$$+2\varepsilon\psi(0)V(r(t),t)+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}[\varepsilon\psi'(x)+\varphi'(x)(1+\varepsilon\partial_{t_0})]V(r(t)-x,t)dx,$$

$$(4.6)\qquad B_f(t,\varepsilon)=2\int_0^t d\tau\int_{r(\tau)}^{r(t)+\sqrt{\frac{\alpha}{\varepsilon}}(t-\tau)} f_x(x,\tau)V(r(t)-x,t-\tau)dx+$$

$$+2\int_0^t f(r(\tau),\tau)(V-E)(r(t)-r(\tau),t-\tau)d\tau-2\int_0^t d\tau\int_{r(\tau)}^\infty f_x(x,\tau)E(r(t)-x,t-\tau)dx,$$

$$G_2(t,\varepsilon)=e^{-t/2\varepsilon}\varphi'\left(r(t)+t\sqrt{\frac{\alpha}{\varepsilon}}\right)+2\int_0^t f(r(\tau),\tau)V(r(t)-r(\tau),t-\tau)d\tau+$$

$$+2\varepsilon\psi(0)V(r(t),t)+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}[\varepsilon\psi'(x)+\varphi'(x)(1+\varepsilon\partial_{t_0})]V(r(t)-x,t)dx-$$

$$-\sqrt{\frac{\varepsilon}{\alpha}}\dot a(t)-2\int_0^t \dot a(\tau)[1+\varepsilon\partial_{t_0}+\varepsilon\dot r(\tau)\partial_{x_0}]V(r(t)-r(\tau),t-\tau)d\tau+$$

$$+2\int_0^t d\tau\int_{r(\tau)}^{r(t)+\sqrt{\frac{\alpha}{\varepsilon}}(t-\tau)} f_x(x,\tau)V(r(t)-x,t-\tau)dx,$$

(4.7) $$B_F(t,\varepsilon) = G_2(t,\varepsilon) + 2\int_0^t V_x(r(t)-r(\tau), t-\tau)w_\varepsilon(\tau)\alpha[1-\varepsilon\dot{r}^2(\tau)/\alpha]d\tau.$$

Now, we note that (see (3.1) and Th.s 3.1-3.4), in order to complete the proof of the convergence of u_ε to u we have to estimate the function

(4.8) $$F(t,\varepsilon) = w_\varepsilon(t)[1-\varepsilon\dot{r}^2(t)/\alpha] - w(t).$$

Moreover, recalling (4.1), (4.2) and setting

$$G(t,\varepsilon) = B(t,\varepsilon) + B_b(t,\varepsilon) + B_c(t,\varepsilon) + B_f(t,\varepsilon) + B_F(t,\varepsilon)\dot{r}(t)\sqrt{\varepsilon/\alpha},$$

we see that the function F introduced in (4.8) satisfies the Volterra integral equation

(4.9) $$F(t,\varepsilon) = G(t,\varepsilon) + \int_0^t H(t,\tau,\varepsilon)F(\tau,\varepsilon)d\tau.$$

Therefore, we can estimate F if we are able to find an upper bound for the solution of (4.11). However, we will show later the following (see Th.s (4.3)-(4.7))

Theorem 4.1 *Under the hypotheses* (2.8) – (2.14)

(4.10) $$|G(t,\varepsilon)| \le K_G\varepsilon^{q_G},$$

where K_G is a constant independent of ε, t and q_G is a positive rational number.

Thus, now, we can state

Theorem 4.2 *If the hypotheses* (2.8) – (2.14) *are satisfied there exists a constant k independent of ε, x_0, t_0, such that*

$$|u_\varepsilon(x_0,t_0) - u(x_0,t_0)| \le k\varepsilon^q,$$

where q is a strictly positive rational number. Therefore, u_ε converges uniformly to u.

Proof. Let us consider the integral equation (4.9). Recalling (2.14) and (2.21) for the kernel $H(t,\tau,\varepsilon)$ defined by (4.2) we have

(4.11) $$|H(t,\tau,\varepsilon)| \le 7r_1/\sqrt{\pi\alpha(t-\tau)}.$$

Therefore, we can apply the results of [1, p.97] and obtain an *a priori* upper bound for Volterra integral equation (4.9). Moreover, from (4.10) easily follows

$$|F(t,\varepsilon)| \le K_F\varepsilon^{q_F},$$

where K_F does not depend on t,ε and $q_F > 0$. Now, we can use this inequality to estimate the last term of (3.1). Indeed, recalling (2.21) we get

$$|\alpha\int_0^t F(\tau,\varepsilon)V(x_0-r(\tau), t_0-\tau)d\tau| \le 4\sqrt{\alpha T/\pi}K_F\varepsilon^{q_F}.$$

This, together with Theorems 3.1-3.4, proves completely the theorem.

Obviously, (4.10) must be proved. This will be done by studying one by one the terms defining G. We begin with $B_f(t,\varepsilon)$ given by (4.6).

Theorem 4.3 *Under the hypotheses* (2.12) – (2.14) *there exist a constant* K_{Bf} *independent of* ε, t *and a strictly positive rational number* p_f *such that* $|B_f(t,\varepsilon)| \leq K_{Bf}\varepsilon^{p_f}$.

Proof. From (4.6), setting $Z_2 = r(t) + (t-\tau)\sqrt{\alpha/\varepsilon}$,

$$B_{f1} = 2\int_0^t d\tau \int_{r(\tau)}^{Z_2} f_x(x,\tau)(V-E)(r(t)-x, t-\tau)dx,$$

$$B_{f2} = -2\int_0^t d\tau \int_{r(\tau)}^{\infty} f_x(x,\tau)E(r(t)-x, t-\tau)dx,$$

$$B_{f3} = 2\int_0^t f(r(\tau),\tau)(V-E)(r(t)-r(\tau), t-\tau)d\tau,$$

we immediately have

$$B_f = B_{f1} + B_{f2} + B_{f3}. \tag{4.12}$$

We can apply to B_{f1} and B_{f2} the arguments developed for A_{f1} and A_{f2} in Th.3.1 and get

$$|B_{f1}| \leq 8M'_f\left\{\left[\frac{K}{\sqrt{3}} + 40\sqrt{2}\right]\varepsilon + \left[\left(\frac{4}{3}\right)^{1/4} + \frac{1}{e^2}\right]\sqrt{\frac{T\varepsilon}{\pi}}\right\},\ |B_{f2}| \leq 8\sqrt{2}M'_f\varepsilon. \tag{4.13}$$

Finally, we consider B_{f3} and first assume $t \leq \varepsilon^{3/4}$. Then,

$$|B_{f3}| \leq \frac{5M_f}{\sqrt{\pi\alpha}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}} \leq \frac{10M_f}{\sqrt{\pi\alpha}}\varepsilon^{3/8}. \tag{4.14}$$

If, instead, $t > \varepsilon^{3/4}$ we define

$$D_1 = \left\{0 < \tau < t - \varepsilon^{3/4}\ :\ \frac{\varepsilon}{\alpha}\left[\frac{r(t)-r(\tau)}{t-\tau}\right]^2 \leq \frac{1}{4}\right\},$$

$$D_2 = \left\{0 < \tau < t - \varepsilon^{3/4}\ :\ \frac{\varepsilon}{\alpha}\left[\frac{r(t)-r(\tau)}{t-\tau}\right]^2 > \frac{1}{4}\right\},$$

and observe that

$$|B_{f3}| \leq B_{f31} + B_{f32} + B_{f33}, \tag{4.15}$$

with

$$B_{f31} = 2M_f\int_{D_1} |(V-E)(r(t)-r(\tau), t-\tau)|d\tau,$$

$$B_{f32} = 2M_f \int_{D_2} |(V - E)(r(t) - r(\tau), t - \tau)|d\tau,$$

$$B_{f33} = 2M_f \int_{t-\varepsilon^{3/4}}^{t} |(V - E)(r(t) - r(\tau), t - \tau)|d\tau.$$

The integral B_{f33} is estimated as B_{f3} in the previous case

$$B_{f33} \leq \frac{10M_f}{\sqrt{\pi\alpha}}\varepsilon^{3/8}. \tag{4.16}$$

Moreover, as $[r(t) - r(\tau)]^2/\alpha(t - \tau) > (t - \tau)/4\varepsilon$ on D_2, for B_{f32} we obtain

$$B_{f32} \leq \frac{5M_f}{\sqrt{\pi\alpha}} \int_{D_2} \frac{e^{-\frac{t-\tau}{16\varepsilon}}}{\sqrt{t-\tau}} d\tau \leq \frac{20M_f}{\sqrt{\pi\alpha}} \left(\frac{4T\varepsilon}{e}\right)^{1/4}. \tag{4.17}$$

Finally, we discuss B_{f31} using the same arguments applied to A_{f11} in Th. 3.1 and achieve

$$B_{f31} \leq \frac{8M_f}{\sqrt{\pi\alpha}} \left[K\sqrt{\frac{\pi}{3e}} + \left(\frac{4}{3}\right)^{1/4} + \frac{4}{e^2}\right] \varepsilon^{5/8}.$$

From this last inequality and recalling (4.12)-(4.17) we see that the theorem is proved.

Consider, now, $B_b(t, \varepsilon)$ and show

Theorem 4.4 *Under the hypotheses* $(2.8) - (2.11), (2.14)$ *there exist a constant* K_b *independent of* ε, t *and a strictly positive rational number* p_b *such that* $|B_b(t, \varepsilon)| \leq K_b \varepsilon^{p_b}$.

Proof. From (4.4), by defining

$$B_{b1} = 2\int_0^t \dot{a}(\tau)(E - V)(r(t) - r(\tau), t - \tau)d\tau,$$

$$B_{b2} = -\dot{a}(t)\sqrt{\frac{\varepsilon}{\alpha}} - 2\int_0^t \dot{a}(\tau)\varepsilon(\partial_{t_0} + \dot{r}(\tau)\partial_{x_0})V(r(t) - r(\tau), t - \tau)d\tau,$$

we, obviously, have $B_b(t, \varepsilon) = B_{b1} + B_{b2}$. But,

$$|B_{b2}| = 2\varepsilon|\dot{a}(0)V(r(t), t) + \int_0^t \ddot{a}(\tau)V(r(t) - r(\tau), t - \tau)d\tau| \leq M'_a\sqrt{\frac{\varepsilon}{\alpha}} + 4\sqrt{\frac{T}{\pi\alpha}}M''_a\varepsilon.$$

Moreover, B_{b1} can be treated exactly as B_{f3} of the last theorem.

We consider, then, the term $B(t, \varepsilon)$ given by (4.3) and note that, following [3, Th.3.1] we can show

Theorem 4.5 *Under the hypotheses* (2.8) – (2.14) *there exist a constant* K_B *independent of* ε, t *and a strictly positive rational number* p_B *such that* $|B(t,\varepsilon)| \leq K_B \varepsilon^{p_B}$.

An analogous result holds for $B_c(t,\varepsilon)$ defined in (4.5). Indeed, we have

Theorem 4.6 *Under the hypotheses* (2.8) – (2.11), (2.14) *there exists a constant* K_{Bc} *independent of* ε, t *such that* $|B_c(t,\varepsilon)| \leq K_{Bc}\varepsilon^{p_c}$, *where* p_c *is a strictly positive rational number.*

Proof. From (4.5) we obtain $B_c(t,\varepsilon) = B_{c1} + B_{c2}$, with

$$B_{c1} = 2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} [\varphi'(x) - \varphi'(0)](1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx -$$

$$-2\int_0^{\infty} [\varphi'(x) - \varphi'(0)]E(r(t)-x,t)dx +$$

$$+2\varphi'(0)\int_0^{r(t)} [(1+\varepsilon\partial_{t_0})V(r(t)-x,t) - E(r(t)-x,t)]dx,$$

$$B_{c2} = e^{-t/2\varepsilon}\left[\varphi'\left(r(t) + t\sqrt{\frac{\alpha}{\varepsilon}}\right) - \varphi'(0)\right] + 2\varepsilon\psi(0)V(r(t),t) +$$

$$+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} \varepsilon\psi'(x)V(r(t)-x,t)dx,$$

since

$$e^{-\frac{t}{2\varepsilon}} - 2\int_{r(t)}^{\infty} E(r(t)-x,t)dx + 2\int_{r(t)}^{r(t)+t\sqrt{\alpha/\varepsilon}} (1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx = 0.$$

Moreover,

$$|B_{c2}| \leq M''_{\varphi} e^{-\frac{t}{2\varepsilon}}[r(t) + t\sqrt{\alpha/\varepsilon}] + M_{\psi}\sqrt{\varepsilon/\alpha}\, e^{-\frac{t}{2\varepsilon}} e^{\frac{t}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}(\frac{r(t)-x}{t})^2}} +$$

$$+2\varepsilon M'_{\psi}\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}} V(r(t)-x,t)dx \leq 4M''_{\varphi}\sqrt{\alpha\varepsilon} + M_{\psi}\sqrt{\varepsilon/\alpha} + 8\varepsilon M'_{\psi}.$$

Finally, B_{c1} can be estimated as A_{c1} in Th.3.2.

The study of $B_F(t,\varepsilon)$ is given by

Theorem 4.7 *Under the hypotheses* (2.8) – (2.14) *there exist a constant* K_F *independent of* ε, t *and a strictly positive rational number* p_F *such that*

$$|B_F(t,\varepsilon)\dot{r}(t)\sqrt{\varepsilon/\alpha}| \leq K_F\varepsilon^{p_F}. \tag{4.18}$$

Proof. As (4.7) holds we first consider $G_2(t,\varepsilon)$ and note

$$|e^{-t/2\varepsilon}\varphi'(r(t) + t\sqrt{\alpha/\varepsilon})| \leq M'_{\varphi},$$

$$|2\varepsilon\psi(0)V(r(t),t)+2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}\varepsilon\psi'(x)V(r(t)-x,t)dx|\leq M_\psi\sqrt{\varepsilon/\alpha}+8M'_\psi,$$

$$|2\int_0^{r(t)+t\sqrt{\alpha/\varepsilon}}\varphi'(x)(1+\varepsilon\partial_{t_0})V(r(t)-x,t)dx|\leq 18M'_\varphi,$$

$$|\sqrt{\frac{\varepsilon}{\alpha}}\dot{a}(t)+2\int_0^t\dot{a}(\tau)[1+\varepsilon\partial_{t_0}+\varepsilon\dot{r}(\tau)\partial_{x_0}]V(r(t)-r(\tau),t-\tau)d\tau|=$$

$$=|2\varepsilon\dot{a}(0)V(r(t),t)+2\int_0^t[\varepsilon\ddot{a}(\tau)+\dot{a}(\tau)]V(r(t)-r(\tau),t-\tau)d\tau|\leq$$

$$\leq M'_a\sqrt{\varepsilon/\alpha}+2\varepsilon M''_a\sqrt{T/\pi\alpha}+2M'_a\sqrt{T/\pi\alpha},$$

$$|2\int_0^t f(r(\tau),\tau)V(r(t)-r(\tau),t-\tau)d\tau|\leq 2M_f\sqrt{T/\pi\alpha},$$

$$|2\int_0^t d\tau\int_{r(\tau)}^{r(t)+\sqrt{\frac{\alpha}{\varepsilon}}(t-\tau)}f_x(x,\tau)V(r(t)-x,t-\tau)dx|\leq 8M'_fT.$$

Recalling all these results we get

$$|G_2(t,\varepsilon)|\leq 19M'_\varphi+8M'_\psi+8M'_fT+ \tag{4.19}$$

$$+2(M'_a+M''_a)\sqrt{T/\pi\alpha}+(M_\psi+M'_a)\sqrt{\varepsilon/\alpha}.$$

On the other hand from (2.17) we immediately obtain

$$F_1(t,\varepsilon)=G_1(t,\varepsilon)+\int_0^t H_1(t,\tau,\varepsilon)F_1(\tau,\varepsilon)d\tau. \tag{4.20}$$

where we have defined

$$F_1(t,\varepsilon)=w_\varepsilon(t)[1-\varepsilon\dot{r}^2(t)/\alpha],\ G_1(t,\varepsilon)=[1+\dot{r}(t)\sqrt{\varepsilon/\alpha}]G_2(t,\varepsilon),$$

$$H_1(t,\tau,\varepsilon)=[1+\dot{r}(t)\sqrt{\varepsilon/\alpha}]2\alpha V_x(r(t)-r(\tau),t-\tau).$$

The solution F_1 of integral equation (4.20) can be estimated as F in Th.4.1, since

$$|G_1(t,\varepsilon)|\leq 2|G_2(t,\varepsilon)|,|H_1(t,\tau,\varepsilon)|\leq 2|H_2(t,\tau,\varepsilon)|.$$

Therefore, we achieve

$$|F_1(t,\varepsilon)|\leq M_1+M_2\sqrt{\varepsilon}+M_3\varepsilon,$$

with $M_i,(i=1,2,3)$, constants independent of t and ε. Consequently,

$$|\int_0^t H_1(t,\tau,\varepsilon)F_1(\tau,\varepsilon)d\tau|\leq(M_1+M_2\sqrt{\varepsilon}+M_3\varepsilon)14r_1\sqrt{T/\pi\alpha}. \tag{4.21}$$

Considering (2.14), from (4.19) and (4.21) we obtain (4.18).

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STUDIO PRELIMINARE DELLE CROSTE CALCAREE (CALICHE) DELLA CAPITANATA (PUGLIA)*

Nota di CLIMACO A., DANIELE M.C., DI NOCERA S. & ZAMPARELLI V.

Presentata dai Soci TULLIO PESCATORE e BRUNO D'ARGENIO
Adunanza del 16 Novembre 1991

RIASSUNTO: Vengono riportati i risultati di uno studio preliminare condotto sulle croste calcaree, "caliche", ampiamente diffuse in gran parte della Puglia centro - settentrionale. In particolare è stata presa in esame l' area della Capitanata compresa tra il F. Fortore ed il F. Ofanto, dove sono stati riconosciuti con discontinuità più orizzonti calcarei sovrapposti e/o intercalati alle differenti formazioni geologiche costituenti il locale substrato. Le successioni di croste affioranti nell' area pugliese, nonchè le strutture e le tessiture riconosciute al microscopio, sono analoghe a quelle studiate in altre regioni del Mediterraneo (Spagna, Francia, Marocco) e dell' America centrale. Le condizioni di un clima arido - semiarido, e l' evidente attività biologica riconosciuta nella formazione delle croste calcaree, lasciano intuire per le croste calcaree pugliesi un' origine ed evoluzione di tipo pedogenetica.

ABSTRACT: The results of a preliminary study dealing with calcareous crusts, "caliche", widely spread in central - northern Apulia have been reported. In particular, the area of Capitanata between the rivers Fortore and Ofanto has been examined: here, several calcareous horizons superimposed on and/or intercalated in the different geological formations, constituting the local substrate, have been recognized in a discontinuous way. The crust profiles which outcrop in the Apulian area, as well as the structures and the textures recognized under the microscope, are similar to the ones which have been studied in other Mediterranean regions (Spain, France, Morocco) and in Central America. The arid - semiarid climatic conditions and the evident biological activity recognized in the calcareous crust let us infer that Apulian "caliche" have pedological origin and evolution.

PAROLE CHIAVE: Caliche, Paleosuoli, Paleogeografia, Quaternario

1. INTRODUZIONE

In questa nota vengono riportati i primi dati relativi allo studio delle croste calcaree presenti nell' area pugliese, unica regione italiana in cui si registrano significative segnalazioni di tali depositi.

Lo studio delle croste, già note in gran parte dei territori che si affacciano nell' area mediterranea (Francia, Spagna, Nord Africa) può dare utili informazioni sui caratteri essenziali del clima e della morfologia che hanno interessato quest' area durante il Pleistocene sup. e l' Olocene.

La ricerca, in questa fase preliminare, è rivolta essenzialmente ad accertare la distribuzione dei "caliche" sul territorio pugliese ed a definire i caratteri sedimentologici principali delle diverse litofacies affioranti, senza tralasciare gli aspetti morfologici che hanno potuto influenzare localmente la genesi e lo sviluppo di tali depositi.

In particolare, è stata presa in esame l' area della pianura della Capitanata compresa tra il F. Fortore ed il F. Ofanto, dove sono stati riconosciuti più orizzonti calcarei sovrapposti e/o intercalati a differenti formazioni geologiche costituenti il locale substrato.

2. PRECEDENTI CONOSCENZE

I primi studi sulle croste carbonatiche presenti nell' area pugliese sono da riferire a DE DOMINICIS (1920 a, 1920 b) che considera le croste come formazioni attuali, originatesi per coagulazione di soluzioni carbonatiche ascendenti e di pseudosoluzioni colloidali circolanti in un suolo sabbioso.

PANTANELLI (1939) attribuisce le croste al Quaternario inferiore o medio e le considera come depositi di origine chimica in ambiente lagunare o lacustre.

MINIERI (1955) avanza alcune considerazioni in merito all' origine e al significato paleoclimatico delle croste. La genesi di tali depositi sarebbe legata alla concomitanza di due fattori: l' elevata permeabilità delle formazioni incrostate, ed un regime sub-tropicale secco. Ciò avrebbe favorito la risalita delle soluzioni permeanti, la loro evaporazione e la conseguente deposizione del sedimento calcareo.

STAMPANONI (1959) segnala la presenza di "concrezioni calcaree" su conglomerati calabriani alla sommità di superfici terrazzate.

TRICART & CAILLEUX (1969) segnalarono la presenza di "glacis encroutès" nell' Italia meridionale come esempio di paleoforme residuali sub-aride in zone attualmente temperate.

NEBOIT (1975) in uno studio morfologico della Puglia e della Lucania orientale dedica ampio spazio al problema delle croste carbonatiche e per la prima volta vengono riconosciute e descritte due differenti litofacies, la cui denominazione vede applicata la terminologia francese. L'Autore ritiene che le croste calcaree costituiscono un orizzonte legato ad un caratteristico episodio climatico verificatosi tra il tardo Riss- Wurm.

MAGALDI (1983) ha condotto studi micromorfologici sulle croste calcaree provenienti da suoli di tipo mollisols in Puglia e alfisols in Sardegna, e evidenzia che il processo pedologico di formazione differisce tra i due gruppi di suoli. In particolare, secondo MAGALDI, il fenomeno che porta alla formazione delle croste calcaree coinvolge soluzioni sovrassature di carbonato di calcio.

3. INQUADRAMENTO GEOLOGICO-MORFOLOGICO

Il Tavoliere costituisce un ampio settore della regione pugliese prevalentemente pianeggiante e delimitato ad ovest dalla "avanfossa adriatica meridionale" (CASNEDI, CRESCENTI e TONNA, 1984). Le stratigrafie costruite sulla base di perforazioni profonde per ricerche d' acqua e di idrocarburi (CARISSIMO et alii., 1963) hanno evidenziato che la regione è costituita da un basamento calcareo cretaceo, di cui il Gargano e le Murge rappresentano i margini affioranti; i sedimenti miocenici e plio-pleistocenici, in gran parte arenacei ed argillosi, poggiano su tale substrato carbonatico.

Dall' inizio del Quaternario all' Olocene, nell' area si è verificato un progressivo sollevamento, (intervallo III b, IV-V; CIARANFI et alii, 1983) che ha determinato una interruzione della sedimentazione francamente marina. Inoltre, le ultime variazioni del livello del mare legate alle fasi glacioeustatiche (CASNEDI et alii, 1984; D'ALESSANDRO et alii, 1989) hanno rimodellato i tratti più prossimi alla costa conferendo ad essi un profilo regolare blandamente inclinato verso il mare.

4. TERMINOLOGIA

Il termine "caliche" è stato utilizzato per la prima volta da Blake (1902) per indicare un deposito carbonatico che presentava un diverso grado di cementazione. Talvolta, con questo termine sono stati chiamati depositi di natura diversa da quella calcarea, come nel caso delle ghiaie cementate da nitrato di sodio ed altri sali rinvenute in alcune zone del Cile e del Perù (FAY, 1920; KLOCKAM & RANDHOR, 1947).

Per questo motivo GOUDIE (1972) suggerisce di utilizzare il termine di "calcrete" indicando con esso " i materiali formati dalla cementazione e\o alterazione di un suolo preesistente o di una roccia formata da carbonato di calcio".

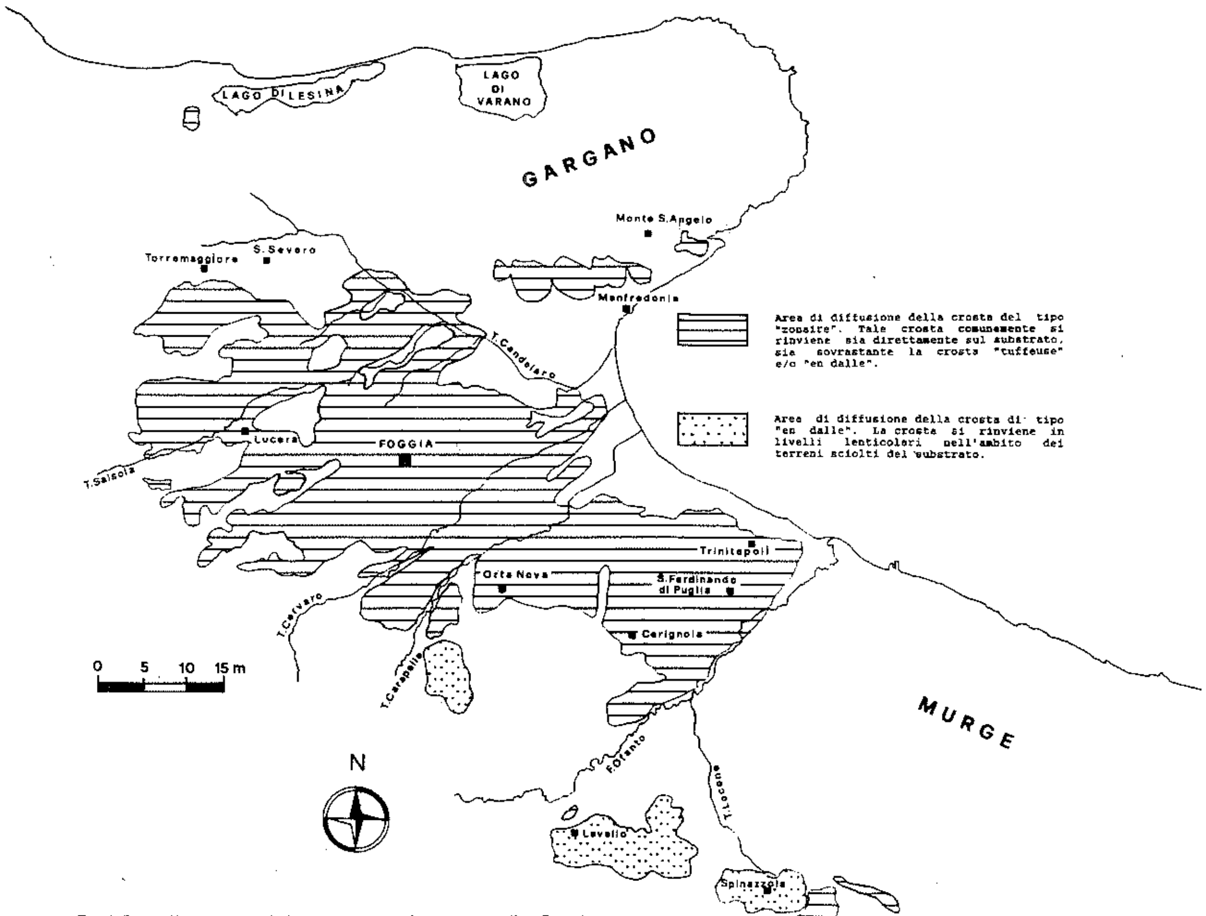

Fig.1: Distribuzione delle croste calcaree nella Capitanata.

In un lavoro fondamentale sulla genesi e distribuzione dei caliche della Spagna e di alcune regioni dell'America centrale ESTEBAN & KLAPPA (1983) propongono nell' ambito dei profili di caliche una suddivisione in quattro diversi orizzonti sovrapposti geometricamente l' uno sull' altro e che dall' alto verso il basso vengono così denominati:

1) hardpan
2) platy
3) nodular-chalky
4) chalky

Altra classificazione delle croste calcaree viene proposta da VOGT (1984) sulla base di studi svolti in alcune regioni della Francia meridionale, Algeria e Marocco. L'Autore distingue i seguenti tipi di croste calcaree:

1) crosta "zonaire" : crosta sommitale molto dura formata dalla sovrapposizione di lamine alternativamente chiare e scure.
2) crosta "tuffeuse" : crosta friabile, a prevalente granulometria fine, il cui strato superficiale si presenta a volte sottilmente laminato ("feuilletee").
3) crosta "en dalle" : crosta con diverso grado di cementazione che ingloba clasti eterometrici e poligenici.

La crosta "zonaire" della scuola francese comprende i due tipi crosta "platy" e "hardpan" di ESTEBAN e KLAPPA (1983); mentre la crosta "tuffeuse" è assimilabile alla crosta "nodular- chalky". Talora, la crosta "platy" è confrontabile anche con la facies "feuilletèe" che il VOGT colloca nella porzione più alta della crosta "tuffeuse" considerandola come termine di transizione tra la crosta "zonaire" e la crosta "tuffeuse".

Per quanto riguarda la crosta "en dalle", essa non trova diretto riscontro nei profili di caliche analizzati da ESTEBAN & KLAPPA, sebbene potrebbe rappresentare un termine di passaggio tra la crosta "chalky" ed il substrato.

In questa fase dello studio delle croste calcaree della Capitanata è stata utilizzata la terminologia della scuola francese, in quanto più aderente alle caratteristiche delle litofacies rinvenute.

5. ANALISI DI CAMPAGNA

Le croste della regione pugliese hanno una distribuzione areale molto più ampia di quanto non venga evidenziato dalla cartografia geologica ufficiale (F. 175 Cerignola, F. 163 Lucera, F. 164 Foggia e F.188 Gravina) (Fig.1). Infatti, tali croste calcaree interessano gran parte della pianura della Capitanata con estensione areale di almeno 3000 kmq. Tale particolare distribuzione è legata indubbiamente alle

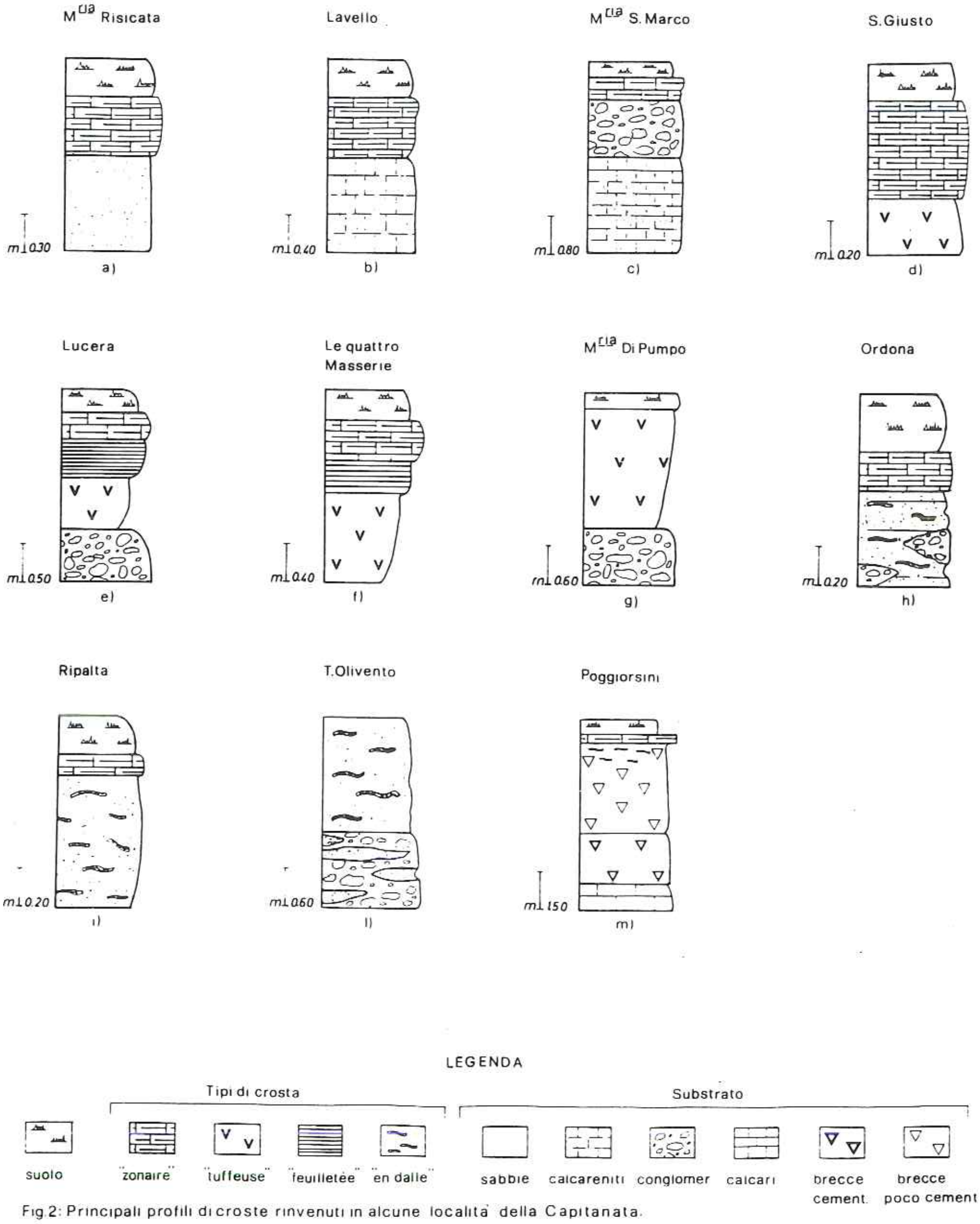

Fig.2: Principali profili di croste rinvenuti in alcune località della Capitanata.

caratteristiche del clima della regione, con escursioni medie annue di poco inferiori ai 17 °C, inverni non eccessivamente rigidi, due massimi di precipitazione a novembre e a marzo, ed un lungo periodo secco estivo (BATTISTA et alii, 1987).

Nell' area della Capitanata, in relazione a locali fattori fisici, geografici e morfologici, si registra localmente un microclima tendente al predesertico che si discosta da quello dominante di tutta la regione pugliese (BISSANTI, 1974; BATTISTA et alii, 1987). Nella regione esaminata i caliche presentano spessori variabili da zona a zona, a partire da qualche decimetro fino a 2,50-3,00 m. Quando gli spessori sono maggiori, si possono distinguere più facies litologiche sovrapposte le une alle altre (Fig.2).

Nell' area compresa tra Cerignola e S. Severo, la crosta di tipo "zonaire", è la più diffusa; ovunque, si presenta ben cementata, con sottili laminazioni, alternativamente di colore chiaro e scuro (Fig.3); lo spessore medio è di circa 0,40m-0,50m. Essa poggia direttamente, sia sui vari terreni del substrato, sia su crosta del tipo "tuffeuse" e/o del tipo "en dalle".

Nei pressi di Cerignola (M.ria Risicata), Lavello ed Ascoli Satriano (M.ria S. Marco), la crosta "zonaire" poggia direttamente sul substrato locale. In loc. M.ria Risicata (fig. 2a) esso è costituito dalle Sabbie di Monte Marano, mentre presso M.ria S. Marco è formato dai conglomerati del ciclo regressivo pleistocenico (Conglomerati di Irsina, Conglomerati di Lamia).

Nei pressi di Lucera, Foggia, Trinitapoli, Orta Nova la crosta "zonaire" evolve inferiormente a crosta "tuffeuse", la quale si presenta ovunque con aspetto pulverulento e solo raramente cementata (Fig.4). In particolare presso Lucera (fig. 2e), dove il substrato locale, in affioramento, è costituito dai conglomerati regressivi pleistocenici, si rinviene un deposito di crosta "tuffeuse" spesso circa 0.50-0.70 m, a cui segue, gradualmente verso l'alto, la crosta "zonaire" con uno spessore di 0.30- 0.40 m. Al passaggio tra questi tipi di crosta è evidente un deposito più cementato e sottilmente laminato (crosta "feuillettée"). Tale situazione si riscontra anche nei pressi di Trinitapoli in località Le Quattro Masserie (fig. 2f) e M.ria Cafiero, ove il substrato non è visibile (Fig.5 e Fig.6).

Presso Torre Maggiore (M.ria di Pumpo, fig. 2g) , dove il substrato è costituito dai depositi alluvionali di età quaternaria appartenenti al I ordine di terrazzi del F. Ofanto, contrariamente alle aree precedenti, si rinviene solo crosta "tuffeuse" con uno spessore di circa 2 m.

In altre località, S. Ferdinando di Puglia e nei pressi di Ordona (fig. 2h), la crosta "zonaire" evolve verso il basso a livelli centimetrici di crosta di tipo "en dalle", scarsamente cementata, priva di laminazione, presente a più altezze nella sottostante formazione delle Sabbie di Monte Marano.

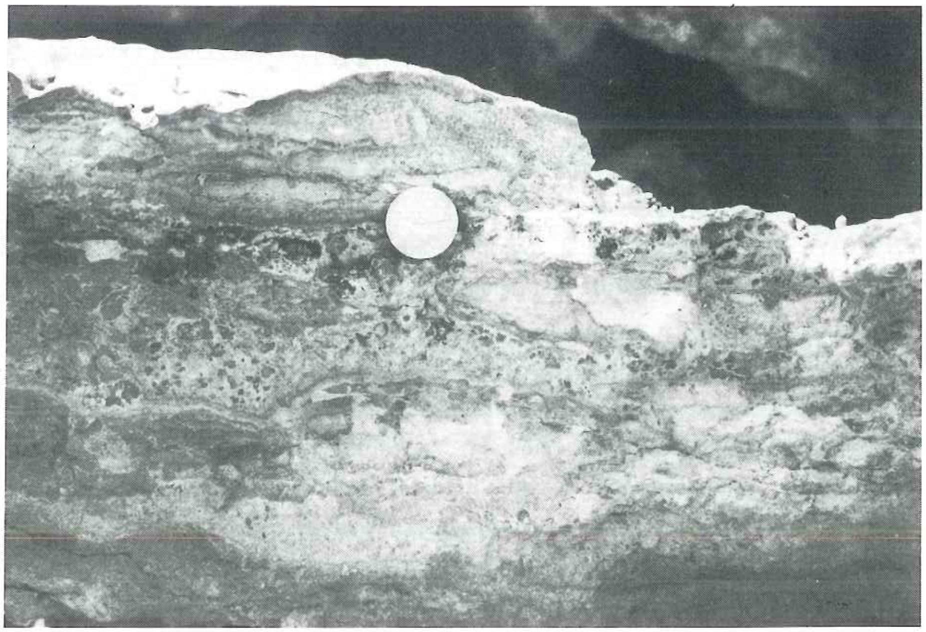

Fig.3 - Aspetto crosta "zonaire". Loc. Posteggio S. Giusto (Troia)

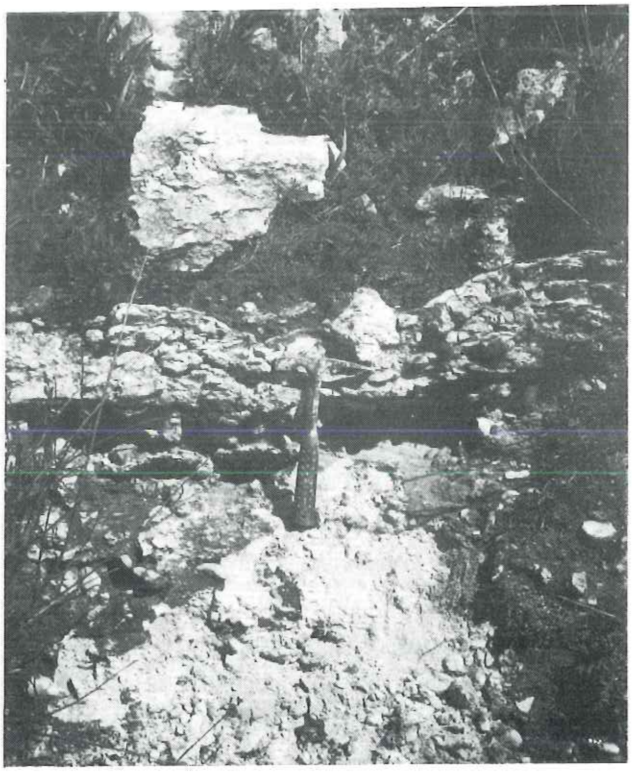

Fig. 4 - Crosta "zonaire" e crosta "tuffeuse" Loc. Posteggio S. Giusto (Troia).

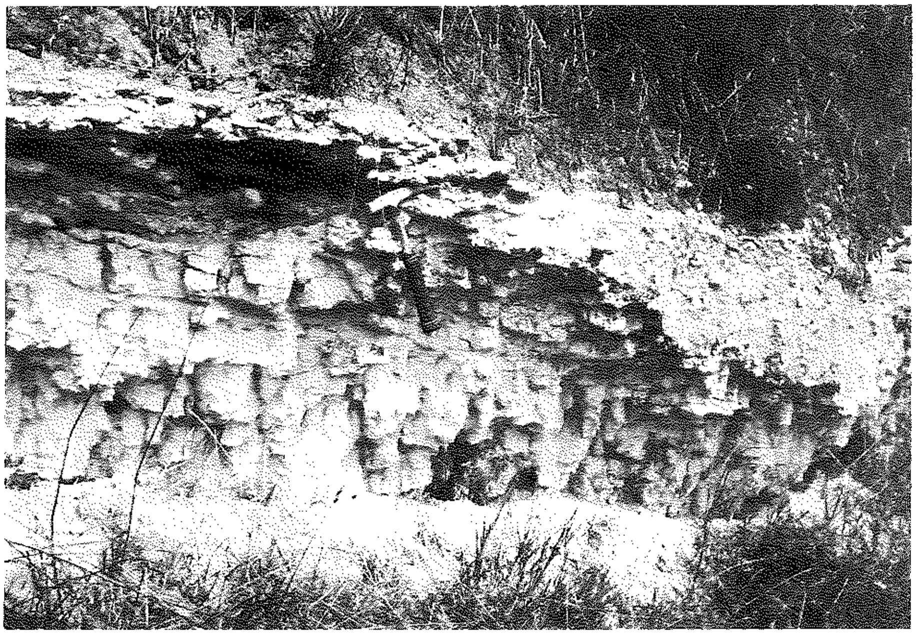

Fig. 5 - "Caliche" affiorante in località le Quattro Masserie; passaggio tra la crosta "zonaire", "feuilletèe" e "tuffeuse".

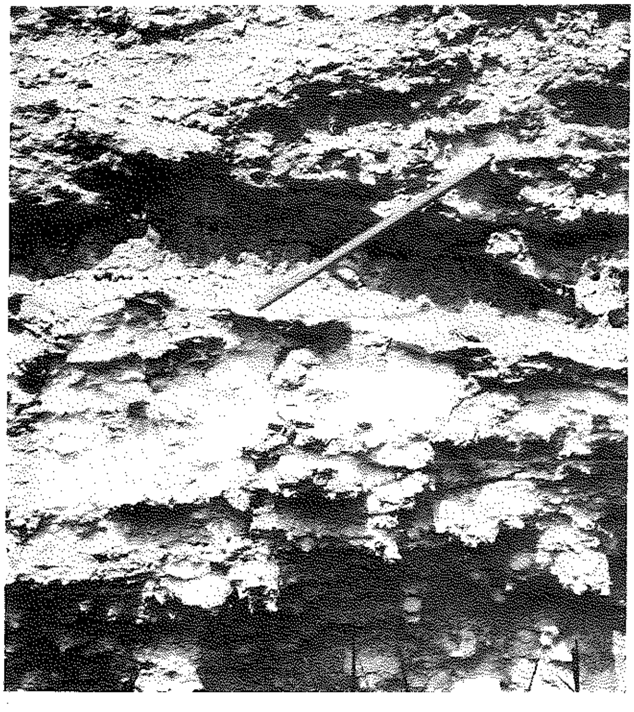

Fig. 6 - Particolare della foto precedente che illustra il passaggio tra la crosta "zonaire" e la crosta "tuffeuse".

Nei dintorni di Ordona livelli discontinui di crosta "en dalle" sono riconoscibili a più altezze nelle sabbie con lenti di puddinghe della formazione dei Conglomerati di Lamia.

Situazioni analoghe sono state riscontrate anche presso Ripalta (fig. 2i), ove il substrato interessato da tali lenti e livelli discontinui calcarei è costituito dalle Sabbie di Serracapriola. Tale fenomeno si registra anche nell'ambito dei depositi clastici di antiche conoidi di deiezione, presenti presso Poggiorsini alle pendici occidentali dei rilievi calcarei delle Murge. Nell'ambito di brecce calcaree, ben cementate, con scarsa matrice, poggianti direttamente sulla formazione dei Calcari di Altamura, a più altezze ed in maniera discontinua si ritrovano livelli lenticolari di concrezioni calcaree di tipo "en dalle", spessi al massimo qualche decimetro. Le concrezioni presentano un colore bianco-rosato con numerose inclusioni di varia natura, quali quarzo, elementi femici e litici. La crosta "zonaire", spessa circa 10-20 cm, sigilla il deposito nella parte alta (fig. 2m).

Anche lungo il T. Olivento (fig. 2 l), dove il substrato è costituito da sabbie con lenti e livelli di conglomerati, sottostanti a sabbie fini di origine vulcanica, si rinvengono intercalati a più altezze, livelli centimetrici di crosta "en dalle" abbastanza cementata.

6. ANALISI DI LABORATORIO

Sono state svolte numerose analisi su campioni prelevati dalle tre litofacies di croste rinvenute in campagna. L' analisi microscopica delle diverse litofacies delle croste pugliesi ha evidenziato vari elementi diagnostici utili per il riconoscimento delle tessiture tipiche dei caliche.

Le tessiture principali riscontrate sono essenzialmente riferibili a quella micritica pelletoidale e a quella alveolare (ESTEBAN, 1974). Nelle diverse matrici sono, inoltre, riconoscibili vari tipi di granuli, principalmente pisoidi e peloidi, strutture tipo rizoliti, fibre di calcite, etc.

La tessitura micritica pelletoidale, è stata riscontrata in tutte le litofacies (ad es. loc. Masseria Cafiero, Masseria Risicata, etc.). Tale tessitura, denominata "clotted" da JAMES (1972), HAY & REEDER (1978), ESTEBAN & KLAPPA (1983), si riconosce fondamentalmente per la presenza di numerosi peloidi, costituiti principalmente da aggregati criptocristallini di calcite, di forma e taglia diversa (Fig.7). In tale tessitura, oltre ai peloidi sono talvolta presenti anche altri granuli tipo pisoidi. Quest'ultimi si differenziano per la presenza di un nucleo, che può essere di varia natura, rivestito da un inviluppo micritico (Fig.8). I peloidi ed i pisoidi, rinvenuti nelle croste della

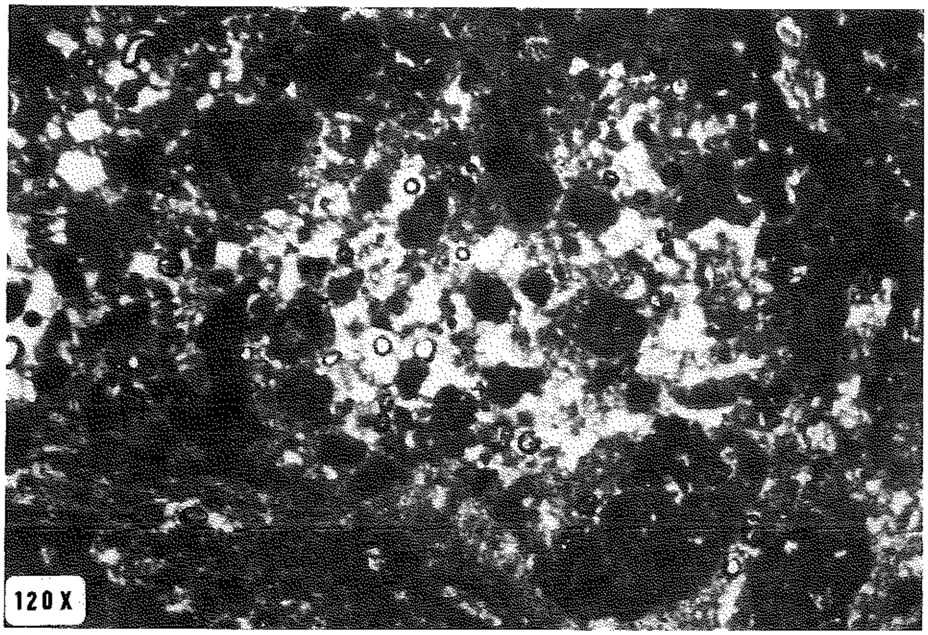

Fig. 7 - Tessitura "clotted" determinata da densi peloidi micritici ("glaebules") (Loc. M.ria Risicata).

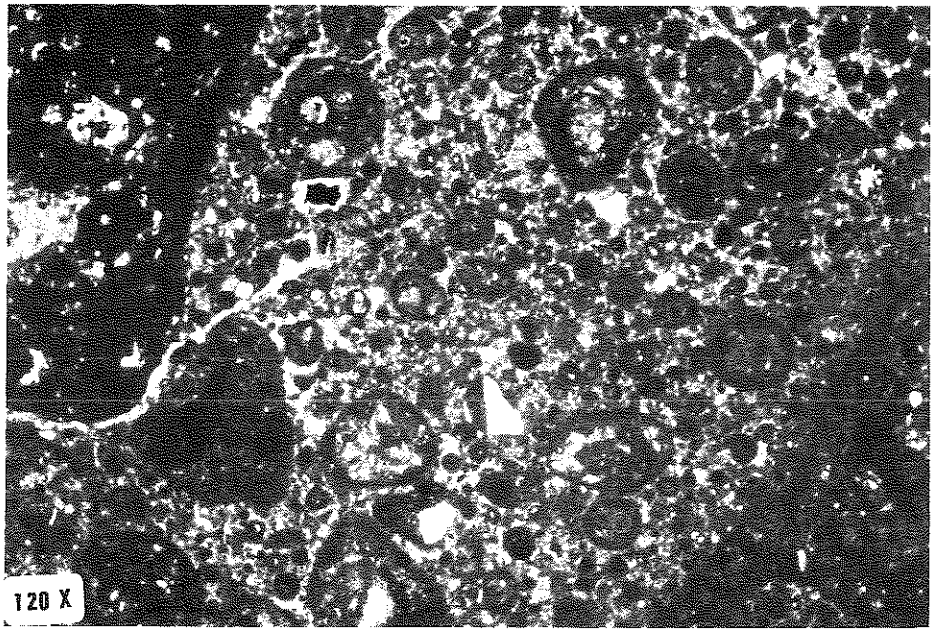

Fig. 8 - "Caliche glaebules" (pisoidi). Granuli formati da nuclei rivestiti da inviluppi micritici poco laminati (Loc. Posteggio S. Giusto).

Capitanata, sono assimilabili ai "glaebules" di ESTEBAN & KLAPPA (1983).

La tessitura alveolare è caratterizzata da pori cilindrici ed irregolari (vuoti primari) separati da pareti o fibre di micrite (Fig.9 e fig.10). Lo spazio tra le "lamine" micritiche è variabile e spesso tra di esse si rinvengono fibre di calcite, che formano delle bande o dei ponti determinando la formazione di pori secondari. Talvolta, questi vuoti sono riempiti da sparite più grossolana, o più comunemente da microsparite o da calcite microcristallina.

I canali e le strutture tipo rizoliti sono comunemente riempite da microsparite (Fig.11). Questa tessitura, con strutture da rizoliti, è molto frequente nella crosta di tipo "zonaire" (loc. Masseria Cafiero), ma è presente anche nelle intercalazioni di crosta di tipo "en dalle" (loc. Torrente Olivento, Poggiorsini) (Fig.12).

Talvolta, nei campioni analizzati si osserva il passaggio tra tessitura micritica pelletoidale e quella alveolare.

L' analisi al microscopio di alcuni campioni di crosta "zonaire" (loc. Masseria Trionfo, Masseria del Bono), ha evidenziato che la tessitura pelletoidale evolve lateralmente o verticalmente ad una tessitura micritica laminata. Le lamine formano delle bande sinuose in cui si riconoscono degli strati micritici alternativamente chiari e scuri. Inoltre, nei campioni di crosta zonaire, le analisi di laboratorio hanno mostrato tra l' altro anche la presenza di frammenti di funghi endolitici.

Le tessiture descritte appaiono analoghe a quelle osservate da diversi autori (ESTEBAN, 1974; CALVET & JULIA, 1983; ESTEBAN & KLAPPA, 1983) in altre aree del Mediterraneo. Tali autori mettono in luce la stretta relazione esistente tra la tessitura alveolare, le radici e le ife funginee, sottolineando in tal modo l' importanza dei processi pedogenetici nella formazione delle croste calcaree. Pertanto, pur non avendo affrontato in questa fase di studio i problemi connessi ai processi di precipitazione e dissoluzione del carbonato di calcio nella zona vadosa, si può ipotizzare sulla base delle osservazioni svolte, che anche per i caliche pugliesi la genesi è stata controllata dai processi pedogenetici.

7. DISCUSSIONE CONCLUSIVA

Le croste calcaree della Pianura della Capitanata si estendono per una vasta area della regione pugliese ampia circa 300.000 ha, e mostrano spessori dell' ordine di 2 - 3m.

L' esame degli affioramenti ha consentito il riconoscimento di varie litofacies con un diverso grado di cementazione che possono talora associarsi formando profili di caliche più o meno completi. Le litofacies riconosciute sono analoghe a quelle segnalate

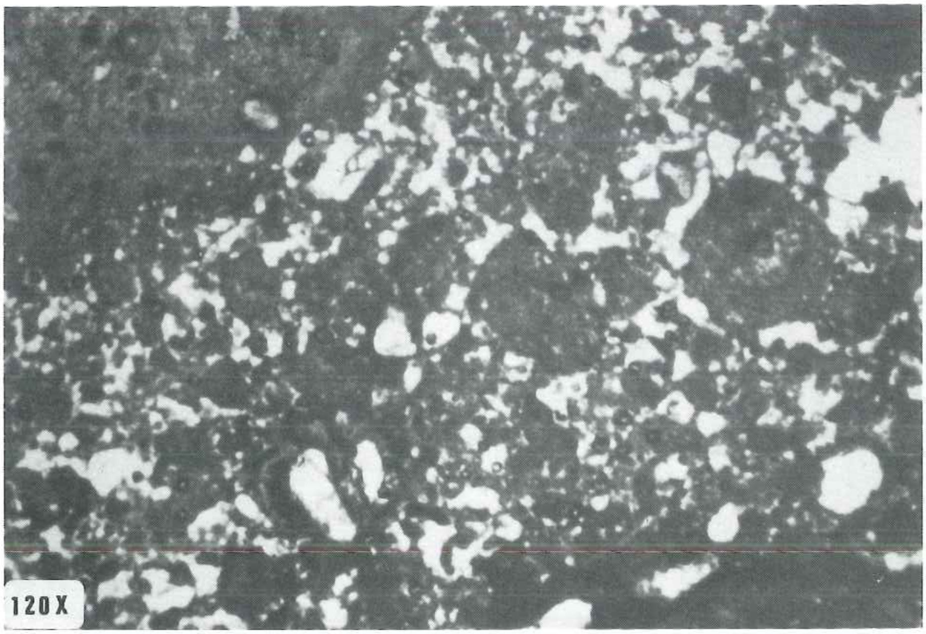

Fig. 9 - Tessitura alveolare. Ponti di micrite che delimitano delle piccole cavità talvolta riempite da cementi; presenti peloidi di taglia variabile (Loc. M.ria Cafiero).

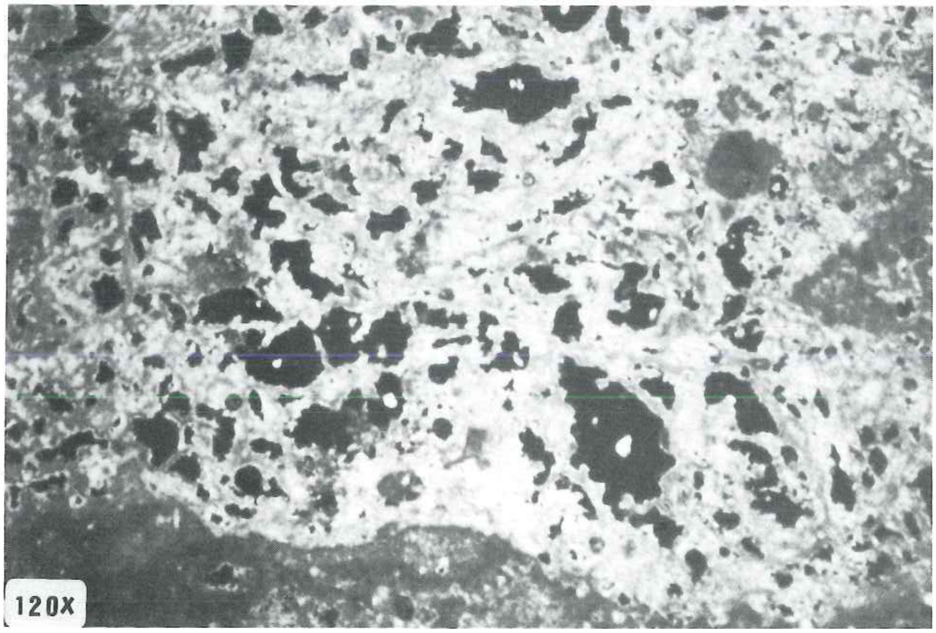

Fig. 10 - Particolare della tessitura alveolare al polarizzatore.

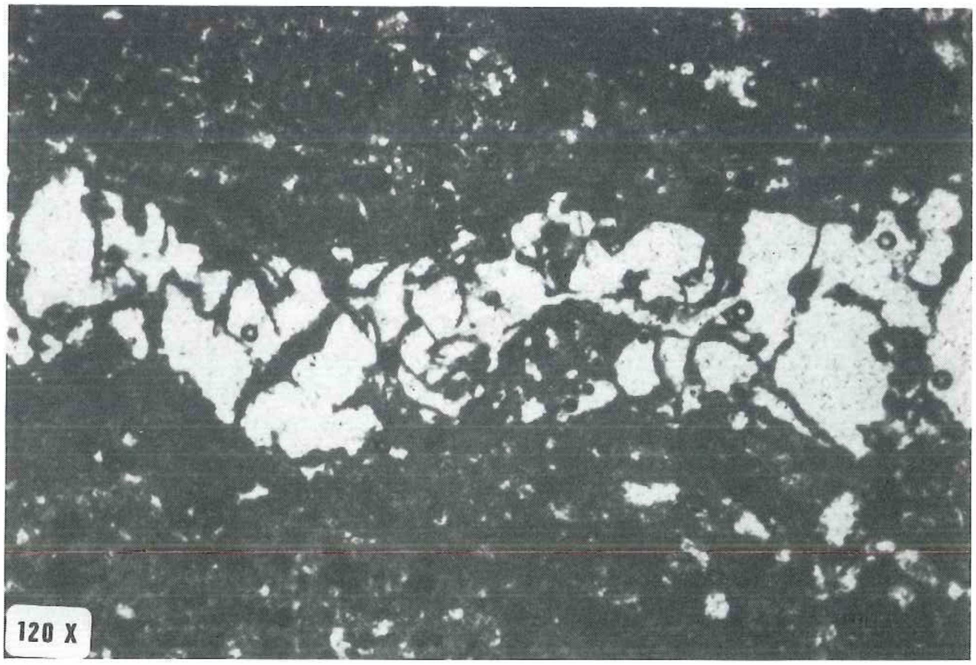

Fig. 11 - Rizolite. Struttura organo-sedimentaria prodotta dall' attività delle radici (Loc. M.ria Cafiero).

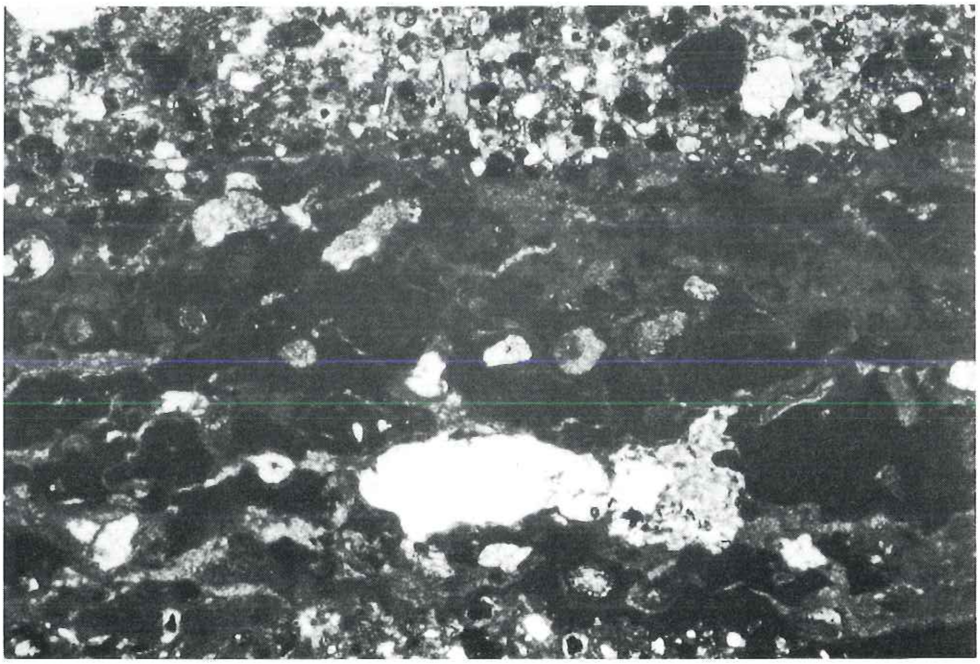

Fig. 12 - Crosta "en dalle" intercalata a sabbie di origine vulcanica (Torrente Olivento).

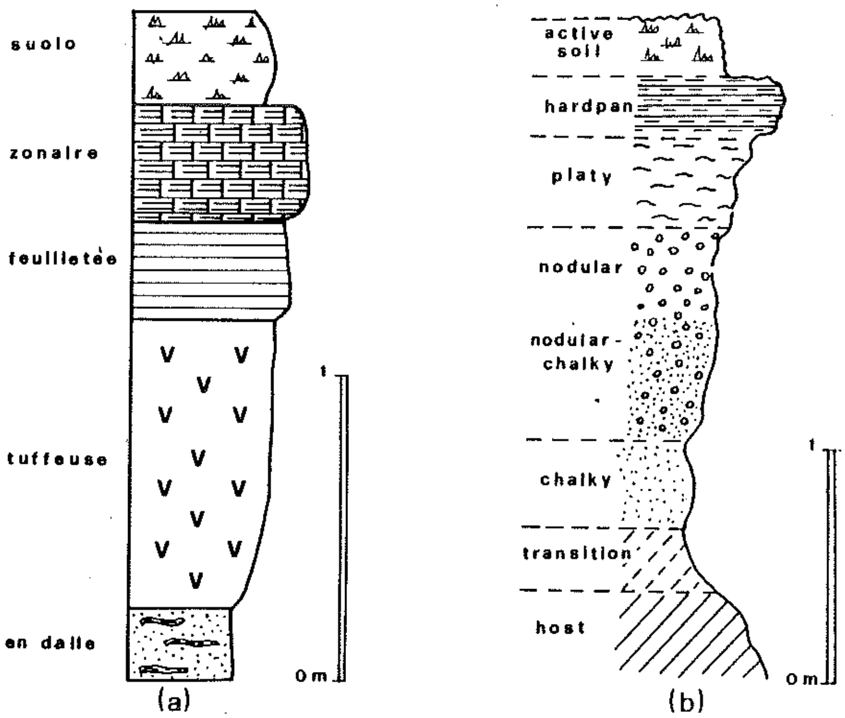

Fig.13: Profili schematici dei «caliche» della Capitanata (a), e di alcune aree della Spagna e dell'America centrale (b, Esteban e Klappa, 1983).

in altre aree del Mediterraneo quali Spagna, Francia, Africa del Nord (ESTEBAN & KLAPPA, 1983; CALVET & JULIA, 1983; VOGT, 1984) (Fig. 13); esse sono :

a) crosta "zonaire" di colore bianco, ben cementata e laminata;

b) crosta "tuffeuse", molto pulverulenta , quasi priva di cementazione;

c) crosta "en dalle", intercalata ai terreni del substrato, di aspetto più massivo e quasi sempre priva di stratificazione.

Anche nella regione pugliese, lo sviluppo delle croste calcaree si realizza sia sul substrato sia nell' ambito di esso; analogamente quasi sempre al top delle facies di "caliche" si rinviene un suolo. L' analisi microscopica ha permesso di riconoscere diverse tessiture. La tessitura alveolare e pelletoidale sono quelle predominanti ed a queste si accompagnano di frequente rizoliti, fibre di calcite e pisoidi.

La presenza di sostanza organica nelle croste carbonatiche sommitali di tipo zonaire, attribuita a resti di funghi endolitici e la particolare struttura alveolare e pelletoidale, dimostrano l' importanza dei microorganismi e delle radici nella formazione dei "caliche". L'attività biologica dei filamenti funginei e/o algali modifica il grado di acidità delle acque favorendo i processi di diagenesi vadosa. Tali processi si sarebbero verificati in condizioni di clima arido-semiarido tendente al pre-desertico. Secondo JAMES & CHOQUETTE (1984) tali particolari condizioni climatiche favoriscono di regola un' ampia distribuzione delle croste calcaree, nonchè lo sviluppo dei profili con più orizzonti e con differente struttura e grado di cementazione. Queste caratteristiche sono largamente rappresentate nei calcreti pugliesi. Sulla base dei fattori esaminati emerge chiaramente che lo sviluppo delle croste carbonatiche della Capitanata si realizza nell' ambito di un substrato vergine (roccia madre)(KLAPPA,1983) favorito da un insieme di processi pedogenetici.

La pedogenesi, controllata dalle condizioni climatiche e morfostrutturali, si sviluppa secondo orizzonti preferenziali ad andamento irregolare. Questi orizzonti dopo un certo tempo si identificano con le diverse litofacies riconosciute, che di regola vengono sigillate dall' orizzonte sommitale della crosta zonaire alla cui formazione possono contribuire sia processi biologici che fisici.

Allo stato attuale dello studio, nei caliche della Capitanata non sono state riconosciute sovrapposizioni di più profili che possano dimostrare il ripetersi di sequenze cicliche determinate da particolari climatiche succedutesi nel corso del quaternario.

L' ampia estensione areale delle croste, nonchè la continuità ed omogeneità delle litofacies, dimostrano che lo sviluppo di tali paleosuoli è stato condizionato, tra l' altro, dalla particolare conformazione morfologica del rilievo molto simile a quella di un pediment ampiamente esteso dal margine orientale della Catena fino all' Adriatico. Lo stato di conservazione dei paleosuoli e del profilo morfologico dell' originario pediment, lasciano intuire che lo sviluppo dei "caliche" della Capitanata si sia realizzato sotto condizioni morfologiche e climatiche non molto differenti dall' attuale ed in un arco di tempo probabilmente compreso tra il Pleistocene sup. e l' Olocene.

Studi più dettagliati estesi ad altre aree, affiancati da analisi morfologiche e strutturali rivolte all' intera regione, potranno chiarire meglio i processi che sono intervenuti nella evoluzione di tali paleosuoli calcarei.

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Di Nocera Silvio, Zamparelli Valeria:

Dipartimento di Scienze della Terra, Facoltà di Scienze, Università di Napoli "Federico II" - Largo San Marcellino, 10 - 80134, Napoli

A NONLINEAR SINGULAR PERTURBATIONS PROBLEM

Nota di Berardino D'Acunto *
Presentata dal Socio Pasquale Renno
Adunanza del 7.12.1991

Riassunto Si studiano delle questioni di perturbazioni singolari con riferimento alle equazioni non lineari del calore e di Cattaneo.
Abstract We discuss some singular perturbations questions related to nonlinear Heat and Cattaneo equations.

1 Introduction

The singular perturbations problems related to *telegraph* and *heat equations* were first discussed by Zlamal in a series of papers, see e.g. [9, 10]. Successively, Fulks and Guenther [5] treated the case of the damped wave equations and Kopáčková-Suchá [6] the mildly nonlinear case.

The physical interest in rigorous approximations of the solutions of the above-mentioted equations is even more important. Indeed, the telegraph equation with a small inertial term was proposed by Cattaneo [1] to replace the classical heat equation in order to remove the paradox of the instantaneous propagation of the thermal disturbance. Moreover, a non-Fourier heat theory can play an important role in many physical situations such as fast flux reactors or laser welding [8].

With this in mind, in some previous works [2, 3] I have already considered hyperbolic-parabolic singular perturbations questions, particularly when the boundary is moving. In this paper the singular perturbations problems for the following equations

$$\varepsilon u_{\varepsilon,tt} + u_{\varepsilon,t} = \alpha u_{\varepsilon,xx} + f(x,t,u_\varepsilon,u_{\varepsilon,x}),$$

$$u_t = \alpha u_{xx} + f(x,t,u,u_x)$$

is discussed with reference, mainly, to the initial values problem.

Following the modern formulations of the singular perturbations problem [4, 7], rigorous approximations estimates are shown. The main result is the following

*Dipartimento di Matematica e Applicazioni, Facoltà di Ingegneria, via Claudio 21, 80125, Napoli.

$$|u_\varepsilon(x,t) - u(x,t)| + |u_{\varepsilon,x}(x,t) - u_x(x,t)| < k\varepsilon^p,$$

where k is a constant independent of ε, x, t and p is a strictly positive rational number. Consequently, we also get the convergence of $(u_\varepsilon, u_{\varepsilon,x})$ to (u, u_x).

2 Statement of the problem

In this paper we study some singular perturbations questions for the following non-linear hyperbolic and parabolic equations

$$\varepsilon u_{\varepsilon,tt} + u_{\varepsilon,t} = \alpha u_{\varepsilon,xx} + f(x,t,u_\varepsilon,u_{\varepsilon,x}), \tag{2.1}$$

$$u_t = \alpha u_{xx} + f(x,t,u,u_x). \tag{2.2}$$

Here, $u_x = \partial u/\partial x$, $u_{\varepsilon,x} = \partial u_\varepsilon/\partial x$.... Moreover, α is the thermal diffusivity and ε the material relaxation time. We first consider the Cauchy problem on

$$D_T = \{(x,t) \ : \ 0 < t \leq T, \ -\infty < x < \infty\}, T > 0, \tag{2.3}$$

with two initial conditions for equation (2.1)

$$u_\varepsilon(x,0) = \phi(x), \ u_{\varepsilon,t}(x,0) = \psi(x), \tag{2.4}$$

and only one, obviously, for (2.2)

$$u(x,0) = \phi(x). \tag{2.5}$$

Later on, we will examine the initial-boundary values problem for the same equations when the following boundary condition is added

$$u_\varepsilon(0,t) = u(0,t) = a(t). \tag{2.6}$$

Under suitable hypotheses (see the end of this section) both the solution of (2.2), (2.5) and the solution of (2.1), (2.4) on D_T can be given by means of integrodifferential equations. So, for problem (2.1), (2.4) we have

$$u_\varepsilon(x,t) = \int_0^t d\tau \int_{x_1(\tau)}^{x_2(\tau)} f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))V(x-\xi,t-\tau)d\xi + \tag{2.7}$$

$$+\frac{e^{-\frac{t}{2\varepsilon}}}{2}[\varphi(x_1(0))+\varphi(x_2(0))] + \int_{x_1(0)}^{x_2(0)} [\varepsilon\psi(\xi)+\varphi(\xi)(1+\varepsilon\partial_t)]V(x-\xi,t)d\xi, \ (x,t) \in D_T.$$

Here, we have introduced the functions

$$x_i(\tau) = x + (-1)^i(t-\tau)\sqrt{\alpha/\varepsilon}, \ i = 1,2, \tag{2.8}$$

and the fundamental solution of (2.1)

$$V(x-\xi,t-\tau)=\frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}}I_0\left(\sqrt{\frac{(t-\tau)^2}{4\varepsilon^2}-\frac{(x-\xi)^2}{4\alpha\varepsilon}}\right), \tag{2.9}$$

where I_n $(n\geq 0)$ is the modified Bessel function of order n.

Similarly, by using the fundamental solution

$$E(x-\xi,t-\tau)=\frac{e^{-\frac{(x-\xi)^2}{4\alpha(t-\tau)}}}{\sqrt{4\pi\alpha(t-\tau)}}, \tag{2.10}$$

the solution of (2.2), (2.5) on D_T is

$$u(x,t)=\int_0^\infty\varphi(\xi)E(x-\xi,t)d\xi+ \tag{2.11}$$

$$+\int_0^t d\tau\int_{-\infty}^\infty f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau))E(x-\xi,t-\tau)d\xi,\ (x,t)\in D_T.$$

By differentiating (2.7), (2.11) with respect to x and setting

$$m(x,t,\varepsilon)=\frac{e^{-\frac{t}{2\varepsilon}}}{2}[\varphi'(x_1(0))+\varphi'(x_2(0))]-\int_0^\infty\varphi'(\xi)E(x-\xi,t)d\xi+ \tag{2.12}$$

$$+\int_{x_1(0)}^{x_2(0)}[\varepsilon\psi'(\xi)+\varphi'(\xi)(1+\varepsilon\partial_t)]V(x-\xi,t)d\xi,$$

$$n(x,t,\varepsilon)=\int_0^t d\tau\int_{x_1(\tau)}^{x_2(\tau)}f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))V_x(x-\xi,t-\tau)d\xi- \tag{2.13}$$

$$-\int_0^t d\tau\int_{-\infty}^\infty f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))E_x(x-\xi,t-\tau)d\xi,$$

$$l(x,t,\varepsilon)=\int_0^t\frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}}[f(x_2(\tau),\tau,u_\varepsilon(x_2(\tau),\tau),u_{\varepsilon,\xi}(x_2(\tau),\tau))- \tag{2.14}$$

$$-f(x_1(\tau),\tau,u_\varepsilon(x_1(\tau),\tau),u_{\varepsilon,\xi}(x_1(\tau),\tau))]d\tau,$$

we obtain

$$u_{\varepsilon,x}(x,t)-u_x(x,t)=m(x,t,\varepsilon)+n(x;t,\varepsilon)+l(x,t,\varepsilon)+ \tag{2.15}$$

$$+\int_0^t d\tau\int_{-\infty}^\infty[f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))-f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau))]E_x(x-\xi,t-\tau)d\xi.$$

Moreover, from (2.7), (2.11), by defining

$$m^*(x,t,\varepsilon)=\frac{e^{-\frac{t}{2\varepsilon}}}{2}[\varphi(x_1(0))+\varphi(x_2(0))]-\int_0^\infty\varphi(\xi)E(x-\xi,t)d\xi+ \tag{2.16}$$

$$+\int_{x_1(0)}^{x_2(0)}[\varepsilon\psi(\xi)+\varphi(\xi)(1+\varepsilon\partial_t)]V(x-\xi,t)d\xi,$$

$$n^*(x,t,\varepsilon)=\int_0^t d\tau\int_{x_1(\tau)}^{x_2(\tau)}f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))V(x-\xi,t-\tau)d\xi- \tag{2.17}$$

$$- \int_0^t d\tau \int_{-\infty}^{\infty} f(\xi, \tau, u_\varepsilon(\xi, \tau), u_{\varepsilon,\xi}(\xi, \tau)) E(x - \xi, t - \tau) d\xi,$$

we get

(2.18) $$u_\varepsilon(x, t) - u(x, t) = m^*(x, t, \varepsilon) + n^*(x, t, \varepsilon) +$$

$$+ \int_0^t d\tau \int_{-\infty}^{\infty} [f(\xi, \tau, u_\varepsilon(\xi, \tau), u_{\varepsilon,\xi}(\xi, \tau)) - f(\xi, \tau, u(\xi, \tau), u_\xi(\xi, \tau))] E(x - \xi, t - \tau) d\xi.$$

We conclude this section with the hypotheses under which we discuss the singular perturbations problem

(2.19) $$\varphi \in C^2(] - \infty, \infty[),$$

(2.20) $|\varphi(x)| < M_\varphi,\ |\varphi'(x)| < M'_\varphi,\ |\varphi''(x)| < M''_\varphi,\ (M_\varphi,\ M'_\varphi,\ M''_\varphi$ const.),

(2.21) $\psi \in C^1(] - \infty, \infty[),\ |\psi(x)| < M_\psi,\ |\psi'(x)| < M'_\psi,\ (M_\psi,\ M'_\psi$ const.),

(2.22) $f \in C^0(\{(x, t, z, w) | (x, t) \in D_T,\ -\infty < z < \infty,\ -\infty < w < \infty\}),$

(2.23) $$|f(x, t, z, w)| < M_f,\ (M_f = \text{const.}),$$

(2.24) $|f(x, t, z, w) - f(x, t, z^*, w^*)| < C_f\{|z - z^*| + |w - w^*|\},\ (C_f = \text{const.}).$

(2.25) for each $C > 0$ and for $|z|, |w| < C$, the function $F(x, t, z, w)$ is uniformly Hölder continuous in x and t for each compact subset of D_T.

Here, obviously, the constants are assumed to be independent of ε.

3 Rigorous approximations. Convergence

We begin the section by proving some rigorous and explicit estimates that we will use in the following.

Theorem 3.1 *Assume hypotheses* (2.19) – (2.25) *are fulfilled. Then one can find a constant K_l independent of ε, x, t and a strictly positive rational number q_l such that*

$$|l(x, t, \varepsilon)| < K_l \varepsilon^{q_l},$$

where l is given by (2.14).

Proof. Indeed we have

$$|l(x, t, \varepsilon)| < M_f \int_0^t \frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{\alpha\varepsilon}} d\tau < 2M_f\sqrt{\varepsilon/\alpha}.$$

Now, we consider $n(x, t, \varepsilon)$ given by (2.13) and show the following

Theorem 3.2 *Assume hypotheses* (2.19) − (2.25) *are fulfilled. Then one can find a constant* K_n *independent of* ε, x, t *and a strictly positive rational number* q_n *such that*

$$|n(x,t,\varepsilon)| < K_n \varepsilon^{q_n}.$$

Proof. By using

$$n_1 = -\int_0^t d\tau \int_{-\infty}^{x_1(\tau)} f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))E_x(x-\xi,t-\tau)d\xi - \tag{3.1}$$

$$-\int_0^t d\tau \int_{x_2(\tau)}^{\infty} f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))E_x(x-\xi,t-\tau)d\xi,$$

$$n_2 = \int_0^t d\tau \int_{x_1(\tau)}^{x_2(\tau)} f(\xi,\tau,u_\varepsilon(\xi,\tau),u_{\varepsilon,\xi}(\xi,\tau))(V_x - E_x)(x-\xi,t-\tau)d\xi - \tag{3.2}$$

with $x_i(\tau)$ defined by (2.8), from (2.13) we have

$$n = n_1 + n_2. \tag{3.3}$$

First, we estimate n_1 and get

$$|n_1| < M_f \int_0^t d\tau \left[\int_{-\infty}^{x_1(\tau)} \frac{(x-\xi)e^{-\frac{(x-\xi)^2}{4\alpha(t-\tau)}}}{4\alpha\sqrt{\pi\alpha}(t-\tau)^{3/2}} + \int_{x_2(\tau)}^{\infty} \frac{(\xi-x)e^{-\frac{(x-\xi)^2}{4\alpha(t-\tau)}}}{4\alpha\sqrt{\pi\alpha}(t-\tau)^{3/2}}\right] d\xi,$$

and, therefore,

$$|n_1| < \frac{M_f}{\sqrt{\pi\alpha}} \int_0^t \frac{e^{-\frac{t-\tau}{4\varepsilon}}}{\sqrt{t-\tau}} d\tau < 2M_f\sqrt{\frac{\varepsilon}{\alpha}}. \tag{3.4}$$

Now, we study n_2 given by (3.2). By setting

$$y_i(\tau) = x + (-1)^i \sqrt{\frac{\alpha}{\varepsilon}} \frac{t-\tau}{2},\ i = 1,2, \tag{3.5}$$

$$n_{21} = M_f \int_0^t d\tau \int_{x_1(\tau)}^{y_1(\tau)} |(V_x - E_x)(x-\xi,t-\tau)| d\xi + \tag{3.6}$$

$$+ M_f \int_0^t d\tau \int_{y_2(\tau)}^{x_2(\tau)} |(V_x - E_x)(x-\xi,t-\tau)| d\xi,$$

$$n_{22} = M_f \int_0^t d\tau \int_{x_1(\tau)}^{x_2(\tau)} |(V_x - E_x)(x-\xi,t-\tau)| d\xi, \tag{3.7}$$

we easily have

$$n_2 < n_{21} + n_{22}. \tag{3.8}$$

We, then, introduce the following inequalities [5, Sec.2]

$$V(x,t) \le 4E(x,t),\ \frac{e^{-\frac{t}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}} \frac{I_1\left(\sqrt{\frac{t^2}{4\varepsilon^2} - \frac{x^2}{4\alpha\varepsilon}}\right)}{\sqrt{1 - \varepsilon x^2/\alpha t^2}} \le 14E(x,t),\ |x| < t\sqrt{\alpha/\varepsilon}, \tag{3.9}$$

and discuss n_{21} as n_1. So, we obtain

(3.10) $$n_{21} \le \frac{15M_f}{\sqrt{\pi\alpha}}\int_0^t \frac{e^{-\frac{t-\tau}{16\varepsilon}}}{\sqrt{t-\tau}}d\tau \le 60M_f\sqrt{\frac{\varepsilon}{\alpha}}.$$

Moreover, for n_{22}, by defining

$$n_{221} = M_f\int_0^t d\tau\int_{y_1(\tau)}^{y_2(\tau)} \frac{|x-\xi|}{2\alpha(t-\tau)}\frac{e^{-\frac{t-\tau}{2\varepsilon}}}{\sqrt{4\alpha\varepsilon}}\left|\frac{I_1\left(\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{x-\xi}{t-\tau}\right)^2}\right)}{\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{x-\xi}{t-\tau}\right)^2}}-\right.$$

$$\left.-\frac{e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}\left(\frac{x-\xi}{t-\tau}\right)^2}}}{\sqrt{\pi(t-\tau)/\varepsilon}\left[1-\frac{\varepsilon}{\alpha}\left(\frac{x-\xi}{t-\tau}\right)^2\right]^{3/4}}\right|d\xi,$$

$$n_{222} = M_f\int_0^t d\tau\int_{y_1(\tau)}^{y_2(\tau)} \frac{|x-\xi|}{2\alpha(t-\tau)}\frac{e^{-\frac{t-\tau}{2\varepsilon}}e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}(\frac{x-\xi}{t-\tau})^2}}}{\sqrt{4\pi\alpha(t-\tau)}}\left|\frac{1}{[1-\frac{\varepsilon}{\alpha}(\frac{x-\xi}{t-\tau})^2]^{3/4}}-1\right|d\xi,$$

$$n_{223} = M_f\int_0^t d\tau\int_{y_1(\tau)}^{y_2(\tau)} \frac{|x-\xi|}{2\alpha(t-\tau)}\left|\frac{e^{-\frac{t-\tau}{2\varepsilon}}e^{\frac{t-\tau}{2\varepsilon}\sqrt{1-\frac{\varepsilon}{\alpha}(\frac{x-\xi}{t-\tau})^2}}}{\sqrt{4\pi\alpha(t-\tau)}}-E(x-\xi,t-\tau)\right|d\xi,$$

we get

(3.11) $$n_{22} < n_{221}+n_{222}+n_{223}.$$

But, n_{221} can be estimated by using the following property of the modified Bessel functions [12]

(3.12) $$|I_n(z)-e^z/\sqrt{2\pi z}| \le K/z,\ K=\text{constant},\ z>0.$$

Thus, since

(3.13) $$\frac{|x-\xi|}{t-\tau} < \sqrt{\frac{\alpha}{4\varepsilon}} \text{ for } y_1(\tau)<\xi<y_2(\tau),$$

we get

(3.14) $$n_{221} \le \frac{M_fK}{3\sqrt{\alpha\varepsilon}}\int_0^t e^{-\frac{t-\tau}{2\varepsilon}}d\tau \le \frac{2M_fK}{3}\sqrt{\varepsilon/\alpha}.$$

Furthermore, by noting that $e^{-z} \le (4/3ez)^{4/3}$, $z \ge 0$, and using (3.13), we obtain

(3.15) $$n_{222} \le \frac{M_f}{\sqrt{\pi\alpha}}\left(\frac{4}{3}\right)^{\frac{3}{4}}\left(\frac{4}{3e}\right)^{\frac{4}{3}}\int_0^t\frac{(2\varepsilon)^{1/3}d\tau}{(t-\tau)^{5/6}} \le \left(\frac{4}{3}\right)^{\frac{3}{4}}\left(\frac{4}{3e}\right)^{\frac{4}{3}}\frac{6M_fT^{\frac{1}{6}}}{\sqrt{\pi\alpha}}(2\varepsilon)^{\frac{1}{3}}.$$

Finally, we evaluate n_{223}. Since $1-e^{-z} \le z$, for $z \ge 0$ and $1-(v/2)-\sqrt{1-v} \le v^2/2$ for $0 \le v < 1$, we obtain

$$n_{223} \le \frac{\varepsilon M_f}{16\alpha^3}\int_0^t d\tau\int_{y_1(\tau}^{y_2(\tau)}\frac{|x-\xi|^5}{(t-\tau)^4}\frac{e^{-\frac{|x-\xi|^2}{4\alpha(t-\tau)}}}{\sqrt{\pi\alpha(t-\tau)}}d\xi.$$

Hence, by using (3.13) and $e^{-z} \le (7/3ez)^{7/3}$, $z \ge 0$, we get

$$(3.16) \quad n_{223} \le M_f \left(\frac{7}{3e}\right)^{\frac{7}{3}} \frac{(2\varepsilon)^{1/3}}{\sqrt{\pi\alpha}} \int_0^t \frac{d\tau}{(t-\tau)^{5/6}} \le 6M_f \left(\frac{7}{3e}\right)^{\frac{7}{3}} \frac{T^{1/6}}{\sqrt{\pi\alpha}} (2\varepsilon)^{1/3}.$$

Considering (3.1)-(3.4), (3.6)-(3.8), (3.10), (3.11), (3.14)-(3.16) we see that the theorem is proved.

Theorem 3.3 *Assume hypotheses* (2.19) – (2.25) *are fulfilled. Then one can find a constant* K_m *independent of* ε, x, t *and a strictly positive rational number* q_m *such that*

$$|m(x,t,\varepsilon)| < K_m \varepsilon^{q_m},$$

where m *is given by* (2.12).

Proof. First, we define

$$(3.17) \qquad m_1 = \int_{x_1(0)}^{x_2(0)} \varepsilon\psi'(\xi) V(x-\xi,t) d\xi,$$

$$(3.18) \qquad m_2 = \frac{e^{-\frac{t}{2\varepsilon}}}{2}[\varphi'(x_1(0)) + \varphi'(x_2(0)) - 2\varphi'(x)],$$

$$(3.19) \qquad m_3 = \int_{x_1(0)}^{x_2(0)} [\varphi'(\xi) - \varphi'(x)](1+\varepsilon\partial_t) V(x-\xi,t) d\xi - $$
$$- \int_{-\infty}^{\infty} [\varphi'(\xi) - \varphi'(x)] E(x-\xi,t) d\xi.$$

Then, from (2.12) we get

$$(3.20) \qquad m(x,t,\varepsilon) = m_1 + m_2 + m_3,$$

since

$$e^{-\frac{t}{2\varepsilon}} - \int_{-\infty}^{\infty} E(x-\xi,t) d\xi + \int_{x_1(0)}^{x_2(0)} (1+\varepsilon\partial_t) V(x-\xi,t) d\xi = 0.$$

We can estimate m_1 by using (3.9). Indeed,

$$(3.21) \qquad |m_1| < 4\varepsilon M'_\psi \int_{x_1(0)}^{x_2(0)} E(x-\xi,t) d\xi \le 4\varepsilon M'_\psi.$$

Now, we consider m_2 and have

$$(3.22) \quad |m_2| \le \frac{e^{-\frac{t}{2\varepsilon}}}{2}[|\varphi'(x_1(0)) - \varphi'(x)| + |\varphi'(x_2(0)) - \varphi'(x)|] < 2M''_\phi \sqrt{\alpha\varepsilon}.$$

Lastly, we examine m_3. By recalling (3.9) from (3.19) we have

$$|m_3| \le 10 \int_{-\infty}^{\infty} |\varphi'(x_1(0)) - \varphi'(x)| E(x-\xi,t) d\xi,$$

$$|m_3| < 20M'_\varphi \int_{-\infty}^{z_1} E(x-\xi,t) d\xi + 20M'_\varphi \int_{z_2}^{\infty} E(x-\xi,t) d\xi + \frac{5M''_\varphi}{\sqrt{\pi\alpha t}} \int_{z_1}^{z_2} |x-\xi| d\xi,$$

with $z_1 = x - \sqrt{t\alpha/\varepsilon^{1/4}}$, $z_2 = x + \sqrt{t\alpha/\varepsilon^{1/4}}$. Hence,

$$|m_3| < \frac{40}{\sqrt{\pi}} M'_\varphi e^{-\frac{1}{8\varepsilon^{1/4}}} \int_0^\infty e^{-y^2/2} dy + 5M''_\varphi \sqrt{\alpha t/\pi\varepsilon^{1/2}},$$

where $y = (\xi - x)/\sqrt{4\alpha t}$. In conclusion,

$$|m_3| < 160\sqrt{2} M'_\varphi \varepsilon^{1/4} + 5M''_\varphi \sqrt{\alpha t/\pi\varepsilon^{1/2}}. \tag{3.23}$$

Thus, if $t < \varepsilon^{3/4}$ then from (3.23) we get

$$|m_3| < 160\sqrt{2} M'_\varphi \varepsilon^{1/4} + 5M''_\varphi \sqrt{\alpha\varepsilon^{1/4}/\pi}.$$

From this last result and from (3.17)-(3.22) we see the theorem is shown. If, instead, $t \geq \varepsilon^{3/4}$, the proof of the estimate for m_3 has to be modified. This can be done by adapting to this case the arguments used in Th.3.2 for n_{22}, so that the theorem can be proved completely.

Theorem 3.4 *Assume hypotheses* (2.19) – (2.25) *are fulfilled. Then one can find a constant H_m independent of ε, x, t and a strictly positive rational number p_m such that*

$$|m^*(x,t,\varepsilon)| < H_m \varepsilon^{p_m},$$

where m^* is given by (2.16).

Proof. The theorem follows without difficulties from known results (see e.g. [3, 5]).

Theorem 3.5 *Assume hypotheses* (2.19) – (2.25) *are fulfilled. Then one can find a constant H_n independent of ε, x, t and a strictly positive rational number p_n such that*

$$|n^*(x,t,\varepsilon)| < H_n \varepsilon^{p_n},$$

where n^* is given by (2.17).

Proof. By considering the definition (2.9) of the fundamental solution V, this proof follows, with obvious modifications, from the one of Th.3.2.

Finally, we can show the *main theorem.*

Theorem 3.6 *Consider the functions $u_\varepsilon(x,t)$, $u(x,t)$ solutions, respectively, of* (2.1), (2.4) *and* (2.2), (2.5). *If hypotheses* (2.19) – (2.25) *are fulfilled there exists a constant k independent of ε, x, t and a strictly positive rational number p such that*

$$|u_\varepsilon(x,t) - u(x,t)| + |u_{\varepsilon,x}(x,t) - u_x(x,t)| < k\varepsilon^p.$$

4 Conclusions

It can be shown that the method introduced above for the Cauchy problem can be used without difficulties also for the singular perturbations initial-boundary values problem (2.1), (2.2), (2.4)- (2.6).

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Proof. Consider formula (2.18) and introduce the norm

$$||u_\varepsilon - u||(t) = sup\{|u_\varepsilon(x,\tau) - u(x,\tau)| + |u_{\varepsilon,x}(x,\tau) - u_x(x,\tau)|\}, \tag{3.24}$$

$$0 < \tau < t, \ -\infty < x < \infty.$$

From Th.s 3.4, 3.5 we immediately obtain

$$|u_\varepsilon(x,t) - u(x,t)| < h^* \varepsilon^{q^*} + C_f \int_0^t ||u_\varepsilon - u||(\tau) d\tau, \tag{3.25}$$

where h^* is a constant that does not depend on ε, x, t and q^* is a strictly positive rational number. Moreover, we consider formula (2.15) and use (3.24). Recalling Th.s 3.1-3.3 we achieve

$$|u_{\varepsilon,x}(x,t) - u_x(x,t)| < h\varepsilon^q + C_f \int_0^t \frac{||u_\varepsilon - u||(\tau)}{\sqrt{\pi\alpha(t-\tau)}} d\tau, \tag{3.26}$$

where h is a constant that does not depend on ε, x, t and q is a strictly positive rational number. From (3.25), (3.26) it follows

$$||u_\varepsilon - u||(t) < k_1 \varepsilon^p + C_f \int_0^t ||u_\varepsilon - u||(\tau)\{1 + [\pi\alpha(t-\tau)]^{-1/2}\} d\tau, \tag{3.27}$$

where k_1 is a constant that does not depend on ε, x, t and p is a strictly positive rational number. Now, we consider the following integral equation

$$y(t) = k_1 \varepsilon^p + C_f \int_0^t y(\tau)\{1 + [\pi\alpha(t-\tau)]^{-1/2}\} d\tau, \tag{3.28}$$

and note that the solution is

$$y(t) = \frac{k_1 \varepsilon^p}{2t^{3/2}\sqrt{\pi(C_f + C_f^2/4\alpha}} \int_0^\infty \exp\left(-\frac{\rho^2}{4t} + \frac{C_f \rho}{\sqrt{4\alpha}}\right) \sinh\left(\rho\sqrt{C_f + C_f^2/4\alpha}\right) \rho d\rho.$$

Hence, after some manipulations we achieve

$$y(t) \le 2k_1 \varepsilon^p \exp\left(T\left[\frac{C_f}{\sqrt{4\alpha}} + \sqrt{C_f + \frac{C_f^2}{4\alpha}}\right]^2\right). \tag{3.29}$$

Since the solutions of inequality (3.27) are bounded by the solution of integral equation (3.28) [11], we finally obtain

$$||u_\varepsilon - u||(t) < 2k_1 \varepsilon^p \exp\left(T\left[\frac{C_f}{\sqrt{4\alpha}} + \sqrt{C_f + \frac{C_f^2}{4\alpha}}\right]^2\right).$$

Recalling definition (3.24) we see the theorem is proved.

Remark. Th.3.6 implies also that $(u_\varepsilon, u_{\varepsilon,x})$ converges uniformly to (u, u_x).